

PHYS 480/581
General Relativity

Extra Problems #9

Question 1.

Consider the action of a massive free scalar field ϕ

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} (\nabla_\mu \phi) (\nabla_\nu \phi) - \frac{1}{2} m^2 \phi^2 \right], \quad (1)$$

where m is the mass of the field and g is the determinant of the metric.

(a) Using the definition of the stress-energy tensor,

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}}, \quad (2)$$

compute $T_{\mu\nu}$ for the scalar field ϕ . You may find the identity $\ln(\det M) = \text{Tr}(\ln M)$ useful (where M is a square non-singular matrix).

(b) Show that this stress energy tensor satisfies the dominant energy condition ($T_{\mu\nu} t^\mu t^\nu \geq 0$ and $T_{\mu\nu} T^\nu{}_\lambda t^\mu t^\lambda \leq 0$ for all timelike vectors t^μ), and thus the weak ($T_{\mu\nu} t^\mu t^\nu \geq 0$) and the null energy ($T_{\mu\nu} l^\mu l^\nu \geq 0$ for null vectors l^μ) conditions.