Question 1:

a)

We have

$$S_\phi = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}m^2\phi^2\right]$$

$$\Rightarrow \quad \delta S_\phi = \int d^4x \left[-\frac{1}{2} g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}m^2\phi^2\right]\delta\sqrt{-g} \; - \int d^4x \sqrt{-g}\,\frac{1}{2}(\nabla_\mu\phi)(\nabla_\nu\phi)\,\delta g^{\mu\nu}$$

We know that if we can write this as $\delta S_\phi = \int d^4x \, A_{\mu\nu}\,\delta g^{\mu\nu}$, then $A = \frac{\delta S_\phi}{\delta g^{\mu\nu}}$.

The 2nd term is already in this form, so we just need to write $\delta\sqrt{-g}$ in terms of $\delta g^{\mu\nu}$.

$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\,\delta g$, so we just need δg. We can find δg using $\ln g = \mathrm{Tr}(\ln g_{\mu\nu})$.

$$\ln(g) = \mathrm{Tr}(\ln g_{\mu\nu})$$

$$\delta \ln(g) = \mathrm{Tr}(\delta \ln g_{\mu\nu})$$

$$\frac{1}{g}\,\delta g = g^{\mu\nu}\,\delta g_{\mu\nu}$$

$$\delta g = g\, g^{\mu\nu}\,\delta g_{\mu\nu}$$

$\delta g_{\mu\nu}$ can be quickly found from the definition of the inverse metric

$$\delta^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}$$

$$\Rightarrow \quad 0 = g^{\mu\sigma}\,\delta g_{\sigma\nu} + g_{\sigma\nu}\,\delta g^{\mu\sigma}$$

$$g_{\rho\mu} g^{\mu\sigma}\,\delta g_{\sigma\nu} = -g_{\rho\mu}\, g_{\sigma\nu}\,\delta g^{\mu\sigma}$$

$$\delta g_{\rho\nu} = -g_{\rho\mu}\, g_{\sigma\nu}\,\delta g^{\mu\sigma}$$

Renaming $\rho \leftrightarrow \mu$

$$\delta g_{\mu\nu} = -g_{\mu\rho}\, g_{\sigma\nu}\,\delta g^{\rho\sigma}$$

Putting this all together,

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}\,\delta g$$

$$= \frac{-1}{2\sqrt{-g}}\left[g g^{\mu\nu}\,\delta g_{\mu\nu}\right]$$

$$= \frac{-1}{2\sqrt{-g}}\left[g g^{\mu\nu}\left(-g_{\mu\rho}\,g_{\sigma\nu}\,\delta g^{\rho\sigma}\right)\right]$$

$$\underline{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\;g_{\rho\sigma}\,\delta g^{\rho\sigma}}$$

Now, we plug this into δS_ϕ above.

$$\delta S_\phi = \int d^4x\left[\frac{-1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}m^2\phi^2\right]\delta\sqrt{-g} - \int d^4x\sqrt{-g}\,\frac{1}{2}(\nabla_\mu\phi)(\nabla_\nu\phi)\,\delta g^{\mu\nu}$$

$$= \int d^4x\left[\frac{-1}{2}g^{\rho\sigma}(\nabla_\rho\phi)(\nabla_\sigma\phi) - \frac{1}{2}m^2\phi^2\right]\left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}\right) - \int d^4x\sqrt{-g}\,\frac{1}{2}(\nabla_\mu\phi)(\nabla_\nu\phi)\,\delta g^{\mu\nu}$$

$$= \int d^4x\,\frac{1}{2}\sqrt{-g}\left[\frac{1}{2}g^{\rho\sigma}g_{\mu\nu}(\nabla_\rho\phi)(\nabla_\sigma\phi) - (\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{1}{2}g_{\mu\nu}m^2\phi^2\right]\delta g^{\mu\nu}$$

$$\Rightarrow \underline{\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \frac{1}{2}\sqrt{-g}\left[\frac{1}{2}g^{\rho\sigma}g_{\mu\nu}(\nabla_\rho\phi)(\nabla_\sigma\phi) - (\nabla_\mu\phi)(\nabla_\nu\phi) + \frac{1}{2}g_{\mu\nu}m^2\phi^2\right]}$$

This gives us $T_{\mu\nu}$.

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}$$

$$\boxed{T_{\mu\nu} = -\frac{1}{2}g^{\rho\sigma}g_{\mu\nu}(\nabla_\rho\phi)(\nabla_\sigma\phi) + (\nabla_\mu\phi)(\nabla_\nu\phi) - \frac{1}{2}g_{\mu\nu}m^2\phi^2}$$

b) For timelike vectors t^μ, $g_{\mu\nu} t^\mu t^\nu < 0$.

$$T_{\mu\nu} t^\mu t^\nu = \left[-\frac{1}{2} g^{\rho\sigma} g_{\mu\nu} (\nabla_\rho \phi)(\nabla_\sigma \phi) + (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2} g_{\mu\nu} m^2 \phi^2\right] t^\mu t^\nu$$

$$= -(g_{\mu\nu} t^\mu t^\nu) \frac{1}{2} g^{\rho\sigma} (\nabla_\rho \phi)(\nabla_\sigma \phi) + t^\mu t^\nu (\nabla_\mu \phi)(\nabla_\nu \phi) - (g_{\mu\nu} t^\mu t^\nu) m^2 \phi^2$$

$$= (-g_{\mu\nu} t^\mu t^\nu) \frac{1}{2} (\nabla^\rho \phi)(\nabla_\rho \phi) + [t^\mu \nabla_\mu \phi]^2 + (-g_{\mu\nu} t^\mu t^\nu) m^2 \phi^2$$

In a LOF, $g_{\mu\nu} = \eta_{\mu\nu}$, $t^\mu = (1,0,0,0)$

$$T_{\mu\nu} t^\mu t^\nu = (-\eta_{\mu\nu} t^\mu t^\nu) \frac{1}{2} (\nabla^\rho \phi)(\nabla_\rho \phi) + [t^\mu \nabla_\mu \phi]^2 + (-\eta_{\mu\nu} t^\mu t^\nu) m^2 \phi^2$$

$$= \frac{1}{2} \eta^{\rho\sigma} (\nabla_\rho \phi)(\nabla_\sigma \phi) + [\nabla_t \phi]^2 + m^2 \phi^2$$

$$= \frac{1}{2}\left[-(\nabla_t \phi)^2 + (\nabla_x \phi)^2 + (\nabla_y \phi)^2 + (\nabla_z \phi)^2\right] + [\nabla_t \phi]^2 + m^2 \phi^2$$

$$\boxed{T_{\mu\nu} t^\mu t^\nu = \frac{1}{2}[\nabla_t \phi]^2 + \frac{1}{2}\left[(\nabla_x \phi)^2 + (\nabla_y \phi)^2 + (\nabla_z \phi)^2\right] + m^2 \phi^2 > 0}$$

Now for $T_{\mu\nu} T^{\nu}{}_{\lambda} t^\mu t^\lambda$,

$$T^\nu{}_\lambda = g^{\nu\alpha} T_{\alpha\lambda}$$

$$T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda = g^{\nu\alpha} T_{\mu\nu} T_{\alpha\lambda}\, t^\mu t^\lambda$$

$$= g^{\nu\alpha}\left[-\tfrac12 g^{\rho\sigma} g_{\mu\nu}(\nabla_\rho\phi)(\nabla_\sigma\phi) + (\nabla_\mu\phi)(\nabla_\nu\phi) - \tfrac12 g_{\mu\nu} m^2\phi^2\right]\left[-\tfrac12 g^{\beta\gamma} g_{\alpha\lambda}(\nabla_\beta\phi)(\nabla_\gamma\phi) + (\nabla_\alpha\phi)(\nabla_\lambda\phi) - \tfrac12 g_{\alpha\lambda} m^2\phi^2\right] t^\mu t^\lambda$$

$$= g^{\nu\alpha}\Big[\tfrac14 \underline{g_{\mu\nu} g_{\alpha\lambda}}\, g^{\rho\sigma} g^{\beta\gamma} (\nabla_\rho\phi)(\nabla_\sigma\phi)(\nabla_\beta\phi)(\nabla_\gamma\phi) - \tfrac12 g^{\rho\sigma} g_{\mu\nu}(\nabla_\rho\phi)(\nabla_\sigma\phi)(\nabla_\alpha\phi)(\nabla_\lambda\phi) + \tfrac14 g^{\rho\sigma} \underline{g_{\mu\nu} g_{\alpha\lambda}} (\nabla_\rho\phi)(\nabla_\sigma\phi) m^2\phi^2$$

$g^{\nu\alpha} g_{\mu\nu} g_{\alpha\lambda} = g_{\mu\lambda}$

$$+ (\nabla_\mu\phi)(\nabla_\nu\phi)(\nabla_\alpha\phi)(\nabla_\lambda\phi) - \tfrac12 g^{\beta\gamma} g_{\alpha\lambda}(\nabla_\mu\phi)(\nabla_\nu\phi)(\nabla_\beta\phi)(\nabla_\gamma\phi) - \tfrac12 g_{\alpha\lambda}(\nabla_\mu\phi)(\nabla_\nu\phi) m^2\phi^2$$

$$+ \tfrac14 \underline{g_{\mu\nu} g_{\alpha\lambda}}\, m^4\phi^4 + \tfrac14 \underline{g_{\mu\nu} g_{\alpha\lambda}}\, g^{\beta\gamma}(\nabla_\beta\phi)(\nabla_\gamma\phi) m^2\phi^2 - \tfrac12 g_{\mu\nu}(\nabla_\alpha\phi)(\nabla_\lambda\phi) m^2\phi^2 \Big] t^\mu t^\lambda$$

$$= -\tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)\left[(\nabla^\rho\phi)(\nabla_\rho\phi)\right]^2 - \cancel{\tfrac12(\nabla^\rho\phi)(\nabla_\rho\phi)[t^\mu\nabla_\mu\phi]^2} - \tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)(\nabla^\rho\phi)(\nabla_\rho\phi)m^2\phi^2$$

$$+ \cancel{(\nabla^\rho\phi)(\nabla_\rho\phi)[t^\mu\nabla_\mu\phi]^2} - \cancel{\tfrac12(\nabla^\rho\phi)(\nabla_\rho\phi)[t^\mu\nabla_\mu\phi]^2} - \tfrac12[t^\mu\nabla_\mu\phi]^2 m^2\phi^2$$

$$- \tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)m^4\phi^4 - \tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)(\nabla^\rho\phi)(\nabla_\rho\phi)m^2\phi^2 - \tfrac12[t^\mu\nabla_\mu\phi]^2 m^2\phi^2$$

$$= -\tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)\left[(\nabla^\rho\phi)(\nabla_\rho\phi)\right]^2 - \tfrac12(-g_{\mu\lambda}t^\mu t^\lambda)(\nabla^\rho\phi)(\nabla_\rho\phi)m^2\phi^2 - [t^\mu\nabla_\mu\phi]^2 m^2\phi^2 - \tfrac14(-g_{\mu\lambda}t^\mu t^\lambda)m^4\phi^4$$

we can simplify terms including $A = (\nabla^\rho\phi)(\nabla_\rho\phi)$,

$$\tfrac14 A^2 + \tfrac12 A\, m^2\phi^2 + \tfrac14 m^4\phi^4 = \tfrac14(A+m^2\phi^2)^2$$

$$\boxed{T_{\mu\nu}T^\nu{}_\lambda t^\mu t^\lambda = -\tfrac14(-g_{\mu\nu}t^\mu t^\nu)\left[(\nabla^\rho\phi)(\nabla_\rho\phi) + m^2\phi^2\right]^2 - [t^\mu\nabla_\mu\phi]^2 m^2\phi^2 \le 0}$$