PHYS 480/581
General Relativity

Homework Assignment 10

Due date: Friday 04/05/2024 5pm, submitted electronically on UNM Canvas

Question 1 (4 points).

In class, we have outlined the key steps to derive the Reissner-Nordström solution to the Einstein equation, a spherically symmetric solution around a compact object of mass M and electric charge Q. Using the symmetries of the problem, our starting point was a trial metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The stress-energy tensor appearing on the right-hand side of the Einstein equation was that of electromagnetism

$$T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma}, \tag{2}$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where $E(r)$ is the electric field in the radial direction. Since the trace $T = g^{\mu\nu}T_{\mu\nu}$ of the above stress-energy tensor is zero, the Einstein equation reduces to $R_{\mu\nu} = 8\pi G T_{\mu\nu}$ in this case.

(a) Using the above stress-energy tensor, show that

$$T_{tt} = \frac{E^2(r)}{2B}, \qquad \text{and} \qquad T_{\theta\theta} = \frac{r^2E^2(r)}{2AB}. \tag{4}$$

(b) Using the tt and rr components of the Einstein equation, we showed that $B(r) = 1/A(r)$. Using Maxwell's equation $\nabla_\mu F^{\nu\mu} = 0$, we also showed that

$$E(r) = \frac{Q}{4\pi r^2}, \tag{5}$$

(remember that $c = 1$ here, which automatically sets $\epsilon_0 = \mu_0 = 1$, and results in the electric charge being dimensionless in these units). Use the $\theta\theta$ component of the Einstein equation to show that $A(r)$ obeys the following differential equation

$$\frac{d(rA)}{dr} = 1 - \frac{GQ^2}{4\pi r^2}. \tag{6}$$

(c) Integrating Eq. (6) on both sides and demanding that the metric reduces to the Schwarzschild metric as $Q \to 0$ yields

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{7}$$

which is the desired Reissner-Nordström solution. Since event horizons occur when $g_{tt} = 0$, show that this spacetime has *two* event horizons when $Q^2 < 4\pi GM^2$. Next, show that for $Q^2 > 4\pi GM^2$, no event horizon exists. Black holes with $Q^2 = 4\pi GM^2$ are said to be *extremal*.

Question 2 (2 points).

Moore Problem 10.2

Question 3 (3 points).

Moore Problem 10.9