Physics 480/581
General Relativity

Homework Assignment 10 Solutions

Question 1 (4 points).

In class, we have outlined the key steps to derive the Reissner-Nordström solution to the Einstein equation, a spherically symmetric solution around a compact object of mass M and electric charge Q. Using the symmetries of the problem, our starting point was a trial metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

The stress-energy tensor appearing on the right-hand side of the Einstein equation was that of electromagnetism

$$T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma}, \tag{2}$$

where

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where $E(r)$ is the electric field in the radial direction. Since the trace $T = g^{\mu\nu}T_{\mu\nu}$ of the above stress-energy tensor is zero, the Einstein equation reduces to $R_{\mu\nu} = 8\pi G T_{\mu\nu}$ in this case.

(a) Using the above stress-energy tensor, show that

$$T_{tt} = \frac{E^2(r)}{2B}, \qquad \text{and} \qquad T_{\theta\theta} = \frac{r^2E^2(r)}{2AB}. \tag{4}$$

Solutions:
From the above, we have $F_{tr} = -E(r)$, and we showed in class that $F^{tr} = E(r)/(AB)$. Thus,

$$\begin{aligned} T_{tt} &= -\frac{1}{4}g_{tt}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{t\alpha}F_{t\gamma} \\ &= \frac{1}{4}A(F^{tr}F_{tr} + F^{rt}F_{rt}) + g^{rr}F_{tr}F_{tr} \\ &= \frac{1}{4}A(-\frac{E}{AB}E - \frac{E}{AB}E) + \frac{1}{B}E^2 \\ &= -\frac{1}{2}\frac{E^2}{B} + \frac{E^2}{B} \\ &= \frac{E^2}{2B}. \end{aligned} \tag{5}$$

$$
\begin{aligned}
T_{\theta\theta} &= -\frac{1}{4}g_{\theta\theta}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\theta\alpha}F_{\theta\gamma} \\
&= -\frac{1}{4}r^2(F^{tr}F_{tr} + F^{rt}F_{rt}) + 0 \\
&= -\frac{1}{4}r^2(-\frac{E}{AB}E - \frac{E}{AB}E) \\
&= \frac{r^2E^2}{2AB}. \qquad (6)
\end{aligned}
$$

(b) Using the tt and rr components of the Einstein equation, we showed that $B(r) = 1/A(r)$. Using Maxwell's equation $\nabla_\mu F^{\nu\mu} = 0$, we also showed that

$$E(r) = \frac{Q}{4\pi r^2}, \qquad (7)$$

(remember that $c = 1$ here, which automatically sets $\epsilon_0 = \mu_0 = 1$, and results in the electric charge being dimensionless in these units). Use the $\theta\theta$ component of the Einstein equation to show that $A(r)$ obeys the following differential equation

$$\frac{d(rA)}{dr} = 1 - \frac{GQ^2}{4\pi r^2}. \qquad (8)$$

Solutions:
The $\theta\theta$ component of the Einstein equation is (using Eq. (23.6c) in Moore)

$$
\begin{aligned}
R_{\theta\theta} &= 8\pi G T_{\theta\theta} \\
-\frac{r}{2AB}\frac{dA}{dr} + \frac{r}{2B^2}\frac{dB}{dr} + 1 - \frac{1}{B} &= 8\pi G\frac{r^2E^2}{2AB} \\
-\frac{r}{2}\frac{dA}{dr} - \frac{r}{2}\frac{dA}{dr} + 1 - A &= 4\pi G r^2 E^2 \\
r\frac{dA}{dr} + A &= 1 - 4\pi G r^2 E^2 \\
\frac{d(rA)}{dr} &= 1 - 4\pi G r^2\left(\frac{Q}{4\pi r^2}\right)^2 \\
\frac{d(rA)}{dr} &= 1 - \frac{GQ^2}{4\pi r^2}. \qquad (9)
\end{aligned}
$$

(c) Integrating Eq. (8) on both sides and demanding that the metric reduces to the Schwarzschild metric as $Q \to 0$ yields

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (10)$$

which is the desired Reissner-Nordström solution. Since event horizons occur when $g_{tt} = 0$, show that this spacetime has *two* event horizons when $Q^2 < 4\pi GM^2$. Next, show that for $Q^2 > 4\pi GM^2$, no event horizon exists.
Solutions:
The location of the event horizon(s) is given by solving

$$\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right) = 0 \qquad (11)$$

This is basically a quadratic equation for r

$$r^2 - 2GMr + \frac{GQ^2}{4\pi} = 0, \tag{12}$$

which means that

$$\begin{aligned} r &= \frac{2GM \pm \sqrt{4G^2M^2 - 4GQ^2/4\pi}}{2} \\ &= GM \pm \sqrt{G^2M^2 - GQ^2/4\pi}. \end{aligned} \tag{13}$$

Thus, if $Q^2 < 4\pi GM^2$, the square root is real and there are two event horizons located at $r = GM \pm \sqrt{G^2M^2 - GQ^2/4\pi}$. On the other hand, if $Q^2 > 4\pi GM^2$, the square root is imaginary and there is no physical event horizon. This case is considered to be unphysical since this would mean that the singularity at $r = 0$ is "naked" (i.e. not hidden behind an event horizon). This has lead people to speculate that the maximum electric charge that a black hole can have is $Q^2 = 4\pi GM^2$.

Question 2 (2 points).

Moore Problem 10.2
Solutions:
The proper time for a purely radial trajectory is given by

$$\begin{aligned} \Delta\tau &= \int \sqrt{-ds^2} \\ &= \int \sqrt{\left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2} \\ &= \int dr \sqrt{\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{dr}\right)^2 - \left(1 - \frac{2GM}{r}\right)^{-1}}. \end{aligned} \tag{14}$$

Now, what is dt/dr? We know that the relativistic energy per unit mass e is a constant of motion and is given by

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}, \tag{15}$$

which for a trajectory starting at rest at $r = \infty$ will be $e = 1$ (since $d\tau \to dt$ there). We thus have,

$$\frac{dt}{dr} = \frac{dt}{d\tau}\frac{d\tau}{dr} = \left(1 - \frac{2GM}{r}\right)^{-1}\frac{d\tau}{dr}. \tag{16}$$

Also, since $\ell = 0$ for purely radial motion, Eq. 10.8 in Moore implies that

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{r}. \tag{17}$$

We thus have

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2GM}{r}\right)^{-2}\left(\frac{2GM}{r}\right)^{-1}. \tag{18}$$

Substituting this in the above

$$\begin{aligned}\Delta\tau &= \int dr\sqrt{\left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{2GM}{r}\right)^{-1}-\left(1-\frac{2GM}{r}\right)^{-1}} \\ &= \int_{2GM}^{10GM} dr\sqrt{\frac{r}{2GM}} \\ &= \frac{1}{\sqrt{2GM}}\frac{2}{3}r^{3/2}\bigg|_{2GM}^{10GM} \\ &= \frac{\sqrt{2}GM}{3}\left(10^{3/2}-2^{3/2}\right) \\ &\approx 13.57GM. \end{aligned} \tag{19}$$

Question 3 (3 points).

Moore Problem 10.9
Solutions:

(a) By the definition of the radial coordinate, the circumference of the orbit is $C = 2\pi r$. We thus have $C = 20\pi GM$. For a solar mass, we know that $GM_\odot = 1.477$ km. For $10^6 M_\odot$, this yields

$$C = 20\pi GM = 20\pi 10^6 GM_\odot \approx 9.28\times 10^7\text{km}. \tag{20}$$

(b) For a stable circular orbit, its radius is given by

$$r_c = \frac{6GM}{1-\sqrt{1-12(GM/\ell)^2}}. \tag{21}$$

We can solve this for ℓ

$$\begin{aligned}\sqrt{1-12(GM/\ell)^2} &= 1-\frac{6GM}{r_c} \\ 1-12(GM/\ell)^2 &= \left(1-\frac{6GM}{r_c}\right)^2 \\ (GM/\ell)^2 &= \frac{1-\left(1-\frac{6GM}{r_c}\right)^2}{12} \\ (GM/\ell)^2 &= \frac{r_c^2-(r_c-6GM)^2}{12r_c^2} \\ (GM/\ell)^2 &= \frac{GMr_c-3(GM)^2}{r_c^2} \\ \ell^2 &= \frac{r_c^2 GM}{r_c-3GM}. \end{aligned} \tag{22}$$

Thus,

$$\ell = \left(\frac{(10GM)^2 GM}{10GM - 3GM} \right)^{1/2} = GM \left(\frac{100}{7} \right)^{1/2} = 10^6 GM_\odot \frac{10}{\sqrt{7}} = 5.58 \times 10^6 \text{km}. \tag{23}$$

The effective energy per unit mass for a circular orbit is

$$\tilde{E} = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}. \tag{24}$$

Plugging ℓ given above and $r = 10GM$ yields

$$\tilde{E} \approx -0.0429. \tag{25}$$

(c) Since $\ell = r^2 d\phi/d\tau$, we have

$$\tau = \frac{r_c^2}{\ell} \int_0^{2\pi} d\phi = \frac{2\pi r_c^2}{\ell} = 2\pi r_c \sqrt{\frac{r_c}{GM} - 3} = 20\pi\sqrt{7} GM \approx 2.46 \times 10^8 \text{km}. \tag{26}$$

We can divide by the speed of light $c = 2.99 \times 10^5$ km/s to obtain $\tau \approx 819$ s.