PHYS 480/581
General Relativity

Homework Assignment 11 Solutions

Question 1 (4 points).

Moore Problem 14.8 ab
Solutions:

(a) Since we are given that

$$u = 1 - \sqrt{\frac{2GM}{r}}, \tag{1}$$

then

$$\begin{aligned} \frac{du}{dt} &= -\sqrt{2GM}\left(-\frac{1}{2}\right) r^{-3/2}\frac{dr}{dt} \\ &= \frac{\sqrt{GM}}{\sqrt{2}r^{3/2}}\frac{dr}{dt}. \end{aligned} \tag{2}$$

According to table 14.1, we have

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{r}\right)\sqrt{1 - \frac{1}{e^2}\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{\ell^2}{r^2}\right)}, \tag{3}$$

where we picked the negative sign since the laser is infalling into the black hole. For a radial trajectory, we have $\ell = 0$. The relativistic energy per unit mass e is a constant of motion. Since the laser starts at rest at infinity, we have

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} = \left(1 - \frac{2GM}{\infty}\right)\frac{dt}{dt} = 1, \tag{4}$$

since $d\tau = dt$ for an object at rest at infinity. We thus have

$$\begin{aligned} \frac{du}{dt} &= -\frac{\sqrt{GM}}{\sqrt{2}r^{3/2}}\left(1 - \frac{2GM}{r}\right)\sqrt{\frac{2GM}{r}} \\ &= -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right). \end{aligned} \tag{5}$$

(b) First, let us manipulate the above expression

$$\begin{aligned} \frac{du}{dt} &= -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right) \\ &= -\frac{GM}{r^2}\left(1 - \sqrt{\frac{2GM}{r}}\right)\left(1 + \sqrt{\frac{2GM}{r}}\right) \\ &= -\frac{GMu}{r^2}\left(1 + \sqrt{\frac{2GM}{r}}\right). \end{aligned} \tag{6}$$

Now, near $r = 2GM$, we have

$$\frac{du}{dt} \approx -\frac{GMu}{(2GM)^2}\left(1+\sqrt{\frac{2GM}{2GM}}\right) = -\frac{u}{2GM}. \tag{7}$$

Integrating this yields

$$u(t) = Ce^{-t/(2GM)}, \tag{8}$$

where C is an integration constant. So, the energy of the laser is exponentially redshifted as it gets close to the event horizon.

Question 2 (2 points).

Moore Problem 16.2

Solutions:

(a) The lifetime of a black hole is given by

$$\tau_{\text{life}} = \frac{256\pi^3 k_B^4}{3G\sigma\hbar^4}(GM)^3. \tag{9}$$

Setting $\tau_{\text{life}} = \tau_{\text{U}} = 13.7 \times 10^9$ yrs, we have

$$M = \frac{1}{G}\left(\frac{3G\sigma\hbar^4\tau_{\text{U}}}{256\pi^3 k_B^4}\right)^{1/3} = \frac{1}{G}\left(\frac{G\hbar\tau_{\text{U}}}{5120\pi}\right)^{1/3}, \tag{10}$$

where we have substituted the definition of the Stefan-Boltzmann constant

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3}, \qquad (c = 1). \tag{11}$$

Personally, I like expressing everything in eV units. We have $G^{-1} = M_{\text{pl}}^2/\hbar = 1.22 \times 10^{28}\text{eV}$ and $\tau_{\text{U}}/\hbar = 6.57 \times 10^{32}\ \text{eV}^{-1}$. Also, 1 eV $= 1.78 \times 10^{-36}$ kg. Thus

$$M = M_{\text{pl}}^{4/3}\left(\frac{\tau_{\text{U}}}{5120\pi\hbar}\right)^{1/3} = 9.68 \times 10^{46}\,\text{eV} = 1.72 \times 10^{11}\,\text{kg}. \tag{12}$$

(b) The mass of a black hole that has one second to live is given by Eq. (12) above, but evaluated at $\tau = 1/\hbar$ s $= 1.52 \times 10^{15}\ \text{eV}^{-1}$.

$$M = M_{\text{pl}}^{4/3}\left(\frac{1.52 \times 10^{15}\text{eV}^{-1}}{5120\pi}\right)^{1/3} = 2.28 \times 10^5\,\text{kg}. \tag{13}$$

To convert this to Joules, we have to multiply by c^2,

$$E_{\text{rel}} = Mc^2 = 2.05 \times 10^{22}\,\text{J}. \tag{14}$$

If an atomic bomb releases $\sim 4 \times 10^{14}$ J, then the energy released in its last second will be equivalent to ~ 51 millions atomic bombs!

Question 3 (1 point).

Moore Problem 16.3
Solutions:
Let's use Eq. (16.9) in Moore, with $M = 1$ TeV $= 1/(1.116 \times 10^{54}) M_\odot = 8.96 \times 10^{-55} M_\odot$

$$\tau_{\text{life}} = 2.095 \times 10^{67} \left(\frac{M}{M_\odot}\right)^3 \text{yrs} = 2.095 \times 10^{67} (8.96 \times 10^{-55})^3 \text{yrs} = 4.75 \times 10^{-88}\,\text{s}. \tag{15}$$

This is tiny! These black holes don't live long enough to swallow the Earth.

Question 4 (1 point).

An extremal non-rotating black holes occurs when $Q^2 = 4\pi G M^2$. Show that such an extremal black hole does not Hawking radiate.
Solutions:
As we discussed in class, the Hawking temperature of the black hole is related to its surface gravity κ via

$$k_B T_H = \frac{\hbar\kappa}{2\pi}. \tag{16}$$

For a non-rotating black hole, the surface gravity is given by

$$\kappa = \frac{1}{2}\frac{\partial}{\partial r}(g_{tt})\Big|_{r=r_{\text{EH}}}. \tag{17}$$

For an extremal black hole with $Q^2 = 4\pi G M^2$, g_{tt} is given by

$$g_{tt} = 1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2} = 1 - \frac{2GM}{r} + \frac{G^2M^2}{r^2} = \left(1 - \frac{GM}{r}\right)^2. \tag{18}$$

Also, for $Q^2 = 4\pi G M^2$ we have $r_{\text{EH}} = GM$ (see homework #10). Thus,

$$\begin{aligned}
\kappa &= \frac{1}{2}\frac{\partial}{\partial r}(g_{tt})\Big|_{r=GM} \\
&= \frac{1}{2}2\left(1 - \frac{GM}{r}\right)\left(\frac{GM}{r^2}\right)\Bigg|_{r=GM} \\
&= \left(1 - \frac{GM}{GM}\right)\left(\frac{GM}{(GM)^2}\right) \\
&= 0,
\end{aligned} \tag{19}$$

which immediately implies that the Hawking temperature of a non-rotating extremal black hole is

$$T_H = 0, \tag{20}$$

and the black hole therefore does not Hawking radiate.