PHYS 480/581
General Relativity

Homework Assignment 12

Due date: Friday 4/19/2024 5pm, submitted electronically on UNM Canvas

Question 1 (12 points).

Let's consider a spacetime that is nearly flat up to small perturbations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ contains the small perturbations. We have argued in class that the different components of $h_{\mu\nu}$ can be written as follows

$$h_{00} = -2\Phi \tag{1}$$
$$h_{0i} = w_i \tag{2}$$
$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij} \tag{3}$$

where Ψ encodes the trace of h_{ij}, and s_{ij} is traceless

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij} \tag{4}$$
$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right), \tag{5}$$

and latin indices (e.g., i, j, k, l) represent only spatial components.

(a) Show that the components of the Ricci tensor take the values

$$R_{00} = \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi \tag{6}$$
$$R_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k \tag{7}$$
$$R_{ij} = -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box^2\Psi\delta_{ij} - \Box^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k, \tag{8}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ and $\Box^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

(b) Use the above to compute the the components G_{00}, G_{0j}, and G_{ij} of the Einstein tensor. Then use the Einstein equation to argue that Ψ, Φ and w^k are not dynamical degrees of freedom, but are rather completely constrained (up to boundary conditions at spatial infinity) once $T_{\mu\nu}$ and s_{ij} are known. To do do, first examine the 00 component of the Einstein equation and work your way from there.

(c) In four spacetime dimensions, the spacetime metric $g_{\mu\nu}$ has 10 independent entries. However, we are always free to make a coordinate transformation $x^\mu \to x^\mu - \xi^\mu$ to set four of these entries to zero. Here, ξ^μ is a function of the spacetime coordinates. First, show that under such a coordinate transformation, the functions introduced above transform as follows

$$\Phi \to \Phi + \partial_0 \xi^0 \tag{9}$$

$$w_i \to w_i + \partial_0 \xi^i - \partial_i \xi^0 \tag{10}$$

$$\Psi \to \Psi - \frac{1}{3}\partial_i \xi^i \tag{11}$$

$$s_{ij} \to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k \xi^k \delta_{ij}. \tag{12}$$

(d) Now, let's use these transformation laws to specify a gauge. The transverse gauge is defined by demanding that $\partial_i s^{ij} = 0$ and $\partial_i w^i = 0$. Note that these are four constraints which will set the four components of ξ^μ. Derive the four differential equations that the components of ξ^μ must satisfy to be in this gauge.

(e) Write down the 00, $0j$, and ij components of the Einstein equation in transverse gauge and in vacuum (i.e., $T_{\mu\nu} = 0$). Argue that if we demand that $h_{\mu\nu} \to 0$ at spatial infinity, then the solution to these equations is $\Psi = 0$, $w_j = 0$, and $\Phi = 0$, and leaving only the following differential equation

$$\Box^2 s_{ij} = 0 \tag{13}$$

to be solved. This is often referred to as the transverse-traceless gauge. Given that s_{ij} is traceless ($\delta^{ij} s_{ij} = 0$) and transverse ($\partial_i s^{ij} = 0$), argue that it contains only two independent degrees of freedom. Thus, what we have shown in this problem is that in vacuum, the metric can only have two dynamical degrees of freedom, which satisfy a wave equation. These are the two possible polarizations of the gravitational waves.

(f) Finally, show that our above solution is indeed the transverse-traceless gauge by verifying that it satisfies all the following conditions (as defined in Moore, see Eq. (31.1) and (31.8))

$$h^{TT}_{0\nu} = 0 \tag{14}$$

$$\eta^{\mu\nu} h^{TT}_{\mu\nu} = 0 \tag{15}$$

$$\partial_\mu h^{\mu\nu}_{TT} = 0 \tag{16}$$

$$\Box^2 h^{TT}_{\mu\nu} = 0. \tag{17}$$