PHYS 480/581
General Relativity

Homework Assignment 12 Solutions

Question 1 (12 points).

Let's consider a spacetime that is nearly flat up to small perturbations, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ contains the small perturbations. We have argued in class that the different components of $h_{\mu\nu}$ can be written as follows

$$h_{00} = -2\Phi \tag{1}$$
$$h_{0i} = w_i \tag{2}$$
$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij} \tag{3}$$

where Ψ encodes the trace of h_{ij}, and s_{ij} is traceless

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij} \tag{4}$$
$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right), \tag{5}$$

and latin indices (e.g., i, j, k, l) represent only spatial components.

(a) Show that the components of the Ricci tensor take the values

$$R_{00} = \nabla^2\Phi + \partial_0\partial_k w^k + 3\partial_0^2\Psi \tag{6}$$
$$R_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k \tag{7}$$
$$R_{ij} = -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box^2\Psi\delta_{ij} - \Box^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k, \tag{8}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ and $\Box^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Solutions:

To first order in the metric perturbation $h_{\mu\nu}$, the Ricci tensor is given by $R_{\beta\nu} = \eta^{\alpha\mu}R_{\alpha\beta\mu\nu}$, where

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right). \tag{9}$$

Note that the trace of the metric is $h \equiv \eta^{\mu\nu}h_{\mu\nu} = -h_{00} + \delta^{ij}h_{ij} = 2\Phi - 6\Psi$. Let's compute each component separately.

$$\begin{aligned}
R_{00} &= \frac{\eta^{\alpha\mu}}{2}\left(\partial_0\partial_\mu h_{\alpha 0} + \partial_\alpha\partial_0 h_{0\mu} - \partial_\alpha\partial_\mu h_{00} - \partial_0\partial_0 h_{\alpha\mu}\right)\\
&= \frac{1}{2}\left(\eta^{\alpha\mu}\partial_0\partial_\mu h_{\alpha 0} + \eta^{\alpha\mu}\partial_\alpha\partial_0 h_{0\mu} - \eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{00} - \partial_0^2\eta^{\alpha\mu}h_{\alpha\mu}\right)\\
&= \frac{1}{2}\left(\eta^{00}\partial_0\partial_0 h_{00} + \delta^{ij}\partial_0\partial_j h_{i0} + \eta^{00}\partial_0\partial_0 h_{00} + \delta^{ij}\partial_i\partial_0 h_{0j} + 2\partial^\mu\partial_\mu\Phi - \partial_0^2 h\right)\\
&= \frac{1}{2}\left(2\partial_0^2\Phi + \delta^{ij}\partial_0\partial_j w_i + 2\partial_0^2\Phi + \delta^{ij}\partial_i\partial_0 w_j + 2\partial^\mu\partial_\mu\Phi - \partial_0^2(2\Phi - 6\Psi)\right)
\end{aligned} \tag{10}$$

$$\begin{aligned} R_{00} &= \frac{1}{2}\left(2\partial_0^2\Phi + 2\partial_0\partial_i w^i + 2(-\partial_0^2 + \nabla^2)\Phi + 6\partial_0^2\Psi\right) \\ &= \nabla^2\Phi + \partial_0\partial_i w^i + 3\partial_0^2\Psi. \end{aligned} \tag{11}$$

$$\begin{aligned} R_{0j} &= \frac{\eta^{\alpha\mu}}{2}\left(\partial_0\partial_\mu h_{\alpha j} + \partial_\alpha\partial_j h_{0\mu} - \partial_\alpha\partial_\mu h_{0j} - \partial_0\partial_j h_{\alpha\mu}\right) \\ &= \frac{1}{2}\left(\eta^{\alpha\mu}\partial_0\partial_\mu h_{\alpha j} + \eta^{\alpha\mu}\partial_\alpha\partial_j h_{0\mu} - \partial^\mu\partial_\mu w_j - \partial_0\partial_j h\right) \\ &= \frac{1}{2}\Big(-\partial_0^2 h_{0j} + \delta^{ik}\partial_0\partial_k h_{ij} - \partial_0\partial_j h_{00} + \delta^{ik}\partial_i\partial_j h_{0k} - \partial^\mu\partial_\mu w_j - \partial_0\partial_j(2\Phi - 6\Psi)\Big) \\ &= \frac{1}{2}\Big(-\partial_0^2 w_j + \delta^{ik}\partial_0\partial_k(2s_{ij} - 2\Psi\delta_{ij}) + 2\partial_0\partial_j\Phi + \delta^{ik}\partial_i\partial_j w_k - \partial^\mu\partial_\mu w_j - \partial_0\partial_j(2\Phi - 6\Psi)\Big) \\ &= \frac{1}{2}\Big(\partial_0\partial_k(2s_j^k - 2\Psi\delta_j^k) + \partial_i\partial_j w^i - \nabla^2 w_j + 6\partial_0\partial_j\Psi\Big) \\ &= \partial_0\partial_k {s_j}^k + \frac{1}{2}\partial_j\partial_k w^k - \frac{1}{2}\nabla^2 w_j + 2\partial_0\partial_j\Psi. \end{aligned} \tag{12}$$

$$\begin{aligned} R_{ij} &= \frac{\eta^{\alpha\mu}}{2}\left(\partial_i\partial_\mu h_{\alpha j} + \partial_\alpha\partial_j h_{i\mu} - \partial_\alpha\partial_\mu h_{ij} - \partial_i\partial_j h_{\alpha\mu}\right) \\ &= \frac{1}{2}\left(\eta^{\alpha\mu}\partial_i\partial_\mu h_{\alpha j} + \eta^{\alpha\mu}\partial_\alpha\partial_j h_{i\mu} - \partial^\mu\partial_\mu h_{ij} - \partial_i\partial_j h\right) \\ &= \frac{1}{2}\Big(-\partial_i\partial_0 w_j + \delta^{kl}\partial_i\partial_l h_{kj} - \partial_0\partial_j w_i + \delta^{kl}\partial_k\partial_j h_{il} - \Box^2(2s_{ij} - 2\Psi\delta_{ij}) - \partial_i\partial_j(2\Phi - 6\Psi)\Big) \\ &= -\partial_0\partial_{(i}w_{j)} - \Box^2 s_{ij} + \Box^2\Psi\delta_{ij} + \delta^{kl}\partial_i\partial_l(s_{kj} - \Psi\delta_{kj}) + \delta^{kl}\partial_k\partial_j(s_{il} - \Psi\delta_{il}) - \partial_i\partial_j(\Phi - 3\Psi) \\ &= -\partial_0\partial_{(i}w_{j)} - \Box^2 s_{ij} + \Box^2\Psi\delta_{ij} + \partial_i\partial_k {s^k}_j + \partial_k\partial_j {s_i}^k - \partial_i\partial_j(\Phi - \Psi) \\ &= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box^2\Psi\delta_{ij} - \Box^2 s_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^k. \end{aligned} \tag{13}$$

(b) Use the above to compute the the components G_{00}, G_{0j}, and G_{ij} of the Einstein tensor. Then use the Einstein equation to argue that Ψ, Φ and w^k are not dynamical degrees of freedom, but are rather completely constrained (up to boundary conditions at spatial infinity) once $T_{\mu\nu}$ and s_{ij} are known. To do do, first examine the 00 component of the Einstein equation and work your way from there.
Solutions:
The first element thing we need is the Ricci scalar

$$\begin{aligned} R &= \eta^{\mu\nu}R_{\mu\nu} \\ &= -R_{00} + \delta^{ij}R_{ij} \\ &= -(\nabla^2\Phi + \partial_0\partial_i w^i + 3\partial_0^2\Psi) + \delta^{ij}(-\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box^2\Psi\delta_{ij} - \Box^2 s_{ij} + 2\partial_k\partial_{(i}{s_{j)}}^k) \\ &= -(\nabla^2\Phi + \partial_0\partial_i w^i + 3\partial_0^2\Psi) - \nabla^2(\Phi - \Psi) - \delta^{ij}\partial_0\partial_{(i}w_{j)} + 3\Box^2\Psi + 2\delta^{ij}\partial_k\partial_{(i}{s_{j)}}^k \\ &= -2\nabla^2\Phi - 2\partial_0\partial_i w^i - 6\partial_0^2\Psi + 4\nabla^2\Psi + 2\partial_i\partial_k s^{ik}. \end{aligned} \tag{14}$$

We can then compute the components of the Einstein tensor:

$$\begin{aligned}G_{00} &= R_{00} - \frac{1}{2}\eta_{00}R \\ &= \nabla^2\Phi + \partial_0\partial_i w^i + 3\partial_0^2\Psi + \frac{1}{2}\left(-2\nabla^2\Phi - 2\partial_0\partial_i w^i - 6\partial_0^2\Psi + 4\nabla^2\Psi + 2\partial_k\partial_i s^{ik}\right) \\ &= 2\nabla^2\Psi + \partial_i\partial_k s^{ik}. \end{aligned} \tag{15}$$

$$\begin{aligned}G_{0j} &= R_{0j} - \frac{1}{2}\eta_{0j}R \\ &= R_{0j} \\ &= -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k. \end{aligned} \tag{16}$$

$$\begin{aligned}G_{ij} &= R_{ij} - \frac{1}{2}\eta_{ij}R \qquad (17) \\ &= -\partial_i\partial_j(\Phi - \Psi) - \partial_0\partial_{(i}w_{j)} + \Box^2\Psi\delta_{ij} - \Box^2 s_{ij} + 2\partial_k\partial_{(i}s_{j)}{}^k \\ &\quad - \frac{1}{2}\delta_{ij}\left(-2\nabla^2\Phi - 2\partial_0\partial_k w^k - 6\partial_0^2\Psi + 4\nabla^2\Psi + 2\partial_l\partial_k s^{lk}\right) \\ &= (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) + 2\delta_{ij}\partial_0^2\Psi - \partial_0\partial_{(i}w_{j)} + \delta_{ij}\partial_0\partial_k w^k - \Box^2 s_{ij} - \delta_{ij}\partial_l\partial_k s^{lk} + 2\partial_k\partial_{(i}s_{j)}{}^k. \end{aligned}$$

Now, the 00 component of the Einstein equation is

$$\begin{aligned}G_{00} &= 8\pi G\,T_{00} \\ 2\nabla^2\Psi + \partial_i\partial_k s^{ik} &= 8\pi G\,T_{00} \\ \nabla^2\Psi &= 4\pi G\,T_{00} - \frac{1}{2}\partial_i\partial_k s^{ik}. \end{aligned} \tag{18}$$

This equation for Ψ has no time derivative at all. Thus, once T_{00} and s^{ik} are known, Ψ is entirely determined by the behavior of the former two quantities, up to boundary conditions at spatial infinity. Thus, Ψ is not a dynamical degree of freedom. The $0j$ component of the Einstein equation is

$$\begin{aligned}G_{0j} &= 8\pi G\,T_{0j} \\ -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\Psi + \partial_0\partial_k s_j{}^k &= 8\pi G\,T_{0j} \\ (\delta_{jk}\nabla^2 - \partial_j\partial_k)w^k &= -16\pi G\,T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j{}^k. \end{aligned} \tag{19}$$

This is an equation for w^k, with no time derivative of w^k. So, if we know the stress-energy tensor and s^{ij} (from which we can also determine Ψ), then we know the vector w^k. Thus, the latter is not an independent dynamical degree of freedom. Finally, the ij component of the Einstein equation reads

$$\begin{aligned}G_{ij} &= 8\pi G\,T_{ij} \\ (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) + 2\delta_{ij}\partial_0^2\Psi - \partial_0\partial_{(i}w_{j)} & \\ +\delta_{ij}\partial_0\partial_k w^k - \Box^2 s_{ij} - \delta_{ij}\partial_l\partial_k s^{lk} + 2\partial_k\partial_{(i}s_{j)}{}^k &= 8\pi G\,T_{ij} \end{aligned} \tag{20}$$

Isolating the Ψ term on the left-hand side yields

$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)\Phi = 8\pi G\, T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Psi - 2\delta_{ij}\partial_0^2\Psi + \partial_0\partial_{(i}w_{j)} - \delta_{ij}\partial_0\partial_k w^k + \Box^2 s_{ij} + \delta_{ij}\partial_l\partial_k s^{lk} - 2\partial_k\partial_{(i}s_{j)}{}^k. \tag{21}$$

Once again, this is an equation for Φ with no time derivative. If we know the stress-energy tensor and s^{ij} (from which we can also get Ψ and w^k), we can determine Φ entirely. Thus, Φ is not an independent dynamical degree of freedom.

(c) In four spacetime dimensions, the spacetime metric $g_{\mu\nu}$ has 10 independent entries. However, we are always free to make a coordinate transformation $x^\mu \to x^\mu - \xi^\mu$ to set four of these entries to zero. Here, ξ^μ is a function of the spacetime coordinates. First, show that under such a coordinate transformation, the functions introduced above transform as follows

$$\Phi \to \Phi + \partial_0\xi^0 \tag{22}$$

$$w_i \to w_i + \partial_0\xi^i - \partial_i\xi^0 \tag{23}$$

$$\Psi \to \Psi - \frac{1}{3}\partial_i\xi^i \tag{24}$$

$$s_{ij} \to s_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}\partial_k\xi^k\delta_{ij}. \tag{25}$$

Solutions:
Under a coordinate transformation $x'^\mu = x^\mu - \xi^\mu$, we saw in class that the metric perturbation $h_{\mu\nu}$ transforms as

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{26}$$

We thus have

$$\begin{aligned} h'_{00} &= h_{00} + 2\partial_0\xi_0 \\ -2\Phi' &= -2\Phi + 2\partial_0\xi_0 \\ \Phi' &= \Phi - \partial_0\xi_0 \\ \Phi' &= \Phi + \partial_0\xi^0, \end{aligned} \tag{27}$$

since $\xi^0 = -\xi_0$. This is the transformation for Φ. Now, let's look at the $0i$ component of the metric perturbation

$$\begin{aligned} h'_{0i} &= h_{0i} + \partial_0\xi_i + \partial_i\xi_0 \\ w'_i &= w_i + \partial_0\xi_i - \partial_i\xi^0, \end{aligned} \tag{28}$$

which is the transformation of w_i. Next, let's look at the trace of the spatial part of the metric.

$$\begin{aligned} \delta^{ij}h'_{ij} &= \delta^{ij}\left(h_{ij} + \partial_i\xi_j + \partial_j\xi_i\right) \\ -6\Psi' &= -6\Psi + 2\partial_i\xi^i \\ \Psi' &= \Psi - \frac{1}{3}\partial_i\xi^i. \end{aligned} \tag{29}$$

Finally, let's consider the ij component of the metric perturbation

$$\begin{aligned} h'_{ij} &= h_{ij} + \partial_i \xi_j + \partial_i \xi_i \\ 2s'_{ij} - 2\Psi'\delta_{ij} &= 2s_{ij} - 2\Psi\delta_{ij} + \partial_i \xi_j + \partial_i \xi_i \\ s'_{ij} - \Psi'\delta_{ij} &= s_{ij} - \Psi\delta_{ij} + \frac{1}{2}(\partial_i \xi_j + \partial_i \xi_i) \\ s'_{ij} &= s_{ij} - \delta_{ij}(\Psi - \Psi + \frac{1}{3}\partial_k \xi^k) + \partial_{(i}\xi_{j)} \\ s'_{ij} &= s_{ij} + \partial_{(i}\xi_{j)} - \delta_{ij}\frac{1}{3}\partial_k \xi^k. \end{aligned} \tag{30}$$

(d) Now, let's use these transformation laws to specify a gauge. The transverse gauge is defined by demanding that $\partial_i s^{ij} = 0$ and $\partial_i w^i = 0$. Note that these are four constraints which will set the four components of ξ^μ. Derive the four differential equations that the components of ξ^μ must satisfy to be in this gauge.

Solutions:

First, we would like to choose ξ^μ to set $\partial_i s'^{ij} = 0$. Using Eq. (30), we have

$$\begin{aligned} \partial_i s'^{ij} &= \partial_i s^{ij} + \partial_i \partial^{(i}\xi^{j)} - \delta^{ij}\frac{1}{3}\partial_i \partial_k \xi^k \\ &= \partial_i s^{ij} + \frac{1}{2}\left(\partial_i \partial^i \xi^j + \partial_i \partial^j \xi^i\right) - \frac{1}{3}\partial^j \partial_k \xi^k \\ &= \partial_i s^{ij} + \frac{1}{2}\nabla^2 \xi^j + \frac{1}{2}\partial^j \partial_k \xi^k - \frac{1}{3}\partial^j \partial_k \xi^k \\ &= \partial_i s^{ij} + \frac{1}{2}\nabla^2 \xi^j + \frac{1}{6}\partial^j \partial_k \xi^k \\ &= 0. \end{aligned} \tag{31}$$

This means that that ξ^i must satisfy the following equation

$$\nabla^2 \xi^j + \frac{1}{3}\partial^j \partial_k \xi^k = -2\partial_i s^{ij}, \tag{32}$$

to put the metric in transverse gauge. Note that this is 3 equations (one for each value of j). The value of ξ^0 is determined by demanding that $\partial_i w'^i = 0$. Using Eq. (28) above, we have

$$\begin{aligned} \partial_i w'^i &= \partial_i w^i + \partial_i \partial_0 \xi^i - \partial_i \partial^i \xi^0 \\ &= \partial_i w^i + \partial_i \partial_0 \xi^i - \nabla^2 \xi^0 \\ &= 0. \end{aligned} \tag{33}$$

This implies that ξ^0 must satisfy the following equation

$$\nabla^2 \xi^0 = \partial_i w^i + \partial_i \partial_0 \xi^i, \tag{34}$$

to put the metric in transverse gauge.

(e) Write down the 00, $0j$, and ij components of the Einstein equation in transverse gauge and in vacuum (i.e., $T_{\mu\nu} = 0$). Argue that if we demand that $h_{\mu\nu} \to 0$ at spatial infinity, then the solution to these equations is $\Psi = 0$, $w_j = 0$, and $\Phi = 0$, and leaving only the following differential equation

$$\Box^2 s_{ij} = 0 \tag{35}$$

to be solved. This is often referred to as the transverse-traceless gauge. Given that s_{ij} is traceless ($\delta^{ij}s_{ij} = 0$) and transverse ($\partial_i s^{ij} = 0$), argue that it contains only two independent degrees of freedom. Thus, what we have shown in this problem is that in vacuum, the metric can only have two dynamical degrees of freedom, which satisfy a wave equation. These are the two possible polarizations of the gravitational waves.

Solutions:

Setting $\partial_i s^{ij} = 0$, $\partial_i w^i = 0$, and $T_{\mu\nu} = 0$ in the Einstein equation written above, we get

$$\nabla^2\Psi = 0, \tag{36}$$

$$\nabla^2 w_j - 4\partial_0\partial_j\Psi = 0, \tag{37}$$

$$(\delta_{ij}\nabla^2 - \partial_i\partial_j)(\Phi - \Psi) + 2\delta_{ij}\partial_0^2\Psi - \partial_0\partial_{(i}w_{j)} - \Box^2 s_{ij} = 0. \tag{38}$$

Now, we know that the solution to Laplace's equation cannot have a local maximum or minimum within the domain of interest. Since we are demanding that $h_{\mu\nu} \to 0$ at spatial infinity (which implies that $\Psi \to 0$ there), then the only possible solution to Eq. (36) is that $\Psi = 0$ everywhere since otherwise there will be a local minimum or maximum somewhere. With $\Psi = 0$, Eq. (37) reduces to $\nabla^2 w_j = 0$, which is also Laplace's equation. Since we demand that $w_j \to 0$ at spatial infinity, then again we must have $w_j = 0$ everywhere to avoid having a local maximum or minimum somewhere.

Now, let's look at the trace of Eq. (39), which takes the form

$$2\nabla^2(\Phi - \Psi) + 6\partial_0^2\Psi = 0, \tag{39}$$

since s^{ij} is traceless and $\partial_i w^i = 0$ in this gauge. Substituting $\Psi = 0$ in this again yield a Laplace equation for Ψ, $\nabla^2\Phi = 0$. For the same reason as outlined above, the only possible solution to this equation is $\Phi = 0$. Then, substituting $\Psi = 0$, $w_i = 0$, and $\Phi = 0$ into Eq. (39) yields

$$\Box^2 s_{ij} = 0, \tag{40}$$

as required. Note that this is a wave equation (not Laplace!), and this equation thus implies that s^{ij} is *not* necessarily zero.

Now, s^{ij} is a symmetric 3×3 tensor, which means it contains 6 independent entries. The fact that is it traceless provides an extra condition, leaving 5 independent entries. The transverse condition $\partial_i s^{ij}$ provides 3 extra constraints (one for each value of j), thus leaving only 2 independent entries. Therefore, s^{ij} only contains two independent degrees of freedom.

(f) Finally, show that our above solution is indeed the transverse-traceless gauge by verifying that it satisfies all the following conditions (as defined in Moore, see Eq. (31.1) and (31.8))

$$h_{0\nu}^{TT} = 0 \tag{41}$$

$$\eta^{\mu\nu}h_{\mu\nu}^{TT} = 0 \tag{42}$$

$$\partial_\mu h^{\mu\nu}_{TT} = 0 \tag{43}$$

$$\Box^2 h_{\mu\nu}^{TT} = 0. \tag{44}$$

Solutions:
Using the fact that $\Psi = 0$, $w_i = 0$, and $\Phi = 0$, the metric perturbation $h_{\mu\nu}^{TT}$ can be written as

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{pmatrix}. \tag{45}$$

This obviously satisfies $h_{0\nu}^{TT} = 0$. The trace condition is also satisfied since

$$\begin{aligned} \eta^{\mu\nu} h_{\mu\nu}^{TT} &= \eta^{00} h_{00}^{TT} + \eta^{ij} h_{ij}^{TT} \\ &= 0 + 2\delta^{ij} s_{ij} \\ &= 0, \end{aligned} \tag{46}$$

since s_{ij} is traceless. The transverse condition $\partial_\mu h_{TT}^{\mu\nu} = 0$ is also satisfied since

$$\partial_\mu h_{TT}^{\mu\nu} = \partial_i s^{ij} = 0. \tag{47}$$

Finally, since the s_{ij} are the only nonzero entries of the metric, it is clear that $\Box^2 h_{\mu\nu}^{TT} = 0$ reduces to

$$\Box^2 s_{ij} = 0, \tag{48}$$

and indeed the traceless-transverse gauge we have defined here is identical to that used in Moore.