PHYS 480/581
General Relativity

Homework Assignment 13 Solutions

Question 1 (4 points).

Moore Problem 31.1
Solutions:
We have the trace-reversed gravitational wave perturbation traveling in the z-direction with

$$H^{\mu\nu} = A^{\mu\nu} \cos(k_\alpha x^\alpha), \tag{1}$$

where $k_\alpha = (-\omega, 0, 0, \omega)$ and

$$A^{\mu\nu} = \begin{pmatrix} a & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ a & 0 & 0 & a \end{pmatrix}. \tag{2}$$

(a) We are asked to show that all conditions in Moore 31.4 are satisfied.

- $k_\alpha k^\alpha = 0$:
$$k_\alpha k^\alpha = \eta^{\mu\alpha} k_\mu k_\alpha = \eta^{tt} k_0^2 + \eta^{zz}(k_z)^2 = -(-\omega)^2 + \omega^2 = 0. \tag{3}$$

- $k_\mu A^{\mu\nu} = 0$: In matrix form, we have

$$\begin{aligned} k_\mu A^{\mu\nu} &= (-\omega, 0, 0, \omega) \begin{pmatrix} a & 0 & 0 & a \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ a & 0 & 0 & a \end{pmatrix} \\ &= (-\omega a + \omega a, 0, 0, -\omega a + \omega a) \\ &= (0, 0, 0, 0). \end{aligned} \tag{4}$$

- $A^{\mu\nu} = A^{\nu\mu}$: By inspection, we see that the matrix $A^{\mu\nu}$ is indeed symmetric.

(b) Let's perform a gauge transformation to put $H^{\mu\nu}$ in transverse-traceless gauge, using $\xi^\mu = B^\mu \sin(k_\sigma x^\sigma)$. The matrix $A^{\mu\nu}$ then transforms as

$$A'^{\mu\nu} = A^{\mu\nu} - k^\mu B^\nu - k^\nu B^\mu + \eta^{\mu\nu} k_\alpha B^\alpha. \tag{5}$$

Note that $k_\alpha B^\alpha = -\omega B^t + \omega B^z = \omega(B^z - B^t)$. To put the above matrix in transverse-traceless gauge, we first need to set $A'^{t\nu} = 0$, that is,

$$0 = A^{t\nu} - k^t B^\nu - k^\nu B^t + \eta^{t\nu}\omega(B^z - B^t). \tag{6}$$

Since $A^{tx} = A^{ty} = 0$ and $k_x = k_y = 0$, we must have $B^x = B^y = 0$ to get $A'^{tx} = A'^{ty} = 0$. Now, taking $\nu = t$ in the above equation,

$$\begin{aligned} 0 &= a - \omega B^t - \omega B^t - \omega(B^z - B^t) \\ &= a - \omega(B^t + B^z), \end{aligned} \tag{7}$$

since $k^t = \omega$ (the sign difference is because we've raised an index using $\eta^{tt} = -1$). Taking $\nu = z$ in Eq. (6), we get

$$\begin{aligned} 0 &= a - \omega B^z - \omega B^t \\ &= a - \omega(B^t + B^z), \end{aligned} \tag{8}$$

which is the same condition as in Eq. (7) and so this equation doesn't provide a new constraint on B^μ. The other condition we need to satisfy is that $A'^{\mu\nu}$ is traceless.

$$\begin{aligned} 0 = \eta_{\mu\nu} A'^{\mu\nu} &= \eta_{\mu\nu} A^{\mu\nu} - \eta_{\mu\nu} k^\mu B^\nu - \eta_{\mu\nu} k^\nu B^\mu + \eta_{\mu\nu} \eta^{\mu\nu} k_\alpha B^\alpha \\ &= b + c - \omega(B^z - B^t) - \omega(B^z - B^t) + 4\omega(B^z - B^t) \\ &= b + c + 2\omega(B^z - B^t). \end{aligned} \tag{9}$$

Now add twice Eq. (7) to (9),

$$\begin{aligned} 0 &= b + c - 2\omega B^t + 2a - 2\omega B^t \\ &= b + c + 2a - 4\omega B^t \\ B^t &= \frac{2a + b + c}{4\omega}. \end{aligned} \tag{10}$$

Plus this into Eq. (7)

$$\begin{aligned} 0 &= a - \omega \left(\frac{2a + b + c}{4\omega} + B^z \right) \\ &= a - \left(\frac{2a + b + c + 4\omega B^z}{4} \right) \\ 4\omega B^z &= 4a - b - c - 2a \\ B^z &= \frac{2a - b - c}{4\omega}. \end{aligned} \tag{11}$$

(c) Now, let's use Eq. (5) above to compute the $A'^{\mu\nu}$. Now, by construction, we have $A'^{t\nu} = 0$. Let's look at the spatial components.

$$\begin{aligned} A'^{xx} &= b - k^x B^x - k^x B^x + \eta^{xx} k_\alpha B^\alpha \\ &= b + \omega(B^z - B^t) \\ &= b + \omega \left(\frac{2a - b - c}{4\omega} - \frac{2a + b + c}{4\omega} \right) \\ &= b - \frac{b + c}{2} \\ &= \frac{b - c}{2}. \end{aligned} \tag{12}$$

$$\begin{aligned} A'^{yy} &= c - k^y B^y - k^y B^y + \eta^{yy} k_\alpha B^\alpha \\ &= c + \omega(B^z - B^t) \\ &= c + \omega \left(\frac{2a - b - c}{4\omega} - \frac{2a + b + c}{4\omega} \right) \\ &= c - \frac{b + c}{2} \\ &= \frac{c - b}{2}. \end{aligned} \tag{13}$$

$$\begin{aligned}
A'^{zz} &= a - k^z B^z - k^z B^z + \eta^{zz} k_\alpha B^\alpha \\
&= a - \omega\left(\frac{2a-b-c}{4\omega}\right) - \omega\left(\frac{2a-b-c}{4\omega}\right) + \omega\left(\frac{2a-b-c}{4\omega} - \frac{2a+b+c}{4\omega}\right) \\
&= a - \left(\frac{2a-b-c}{4}\right) - \left(\frac{2a+b+c}{4}\right) \\
&= a - \frac{4a}{4} \\
&= 0.
\end{aligned} \tag{14}$$

The off-diagonal spatial components are zero since $k_x = k_y = 0$ and $B^x = B^y = 0$. We thus have

$$A'^{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b-c & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

(d) From looking at the above $A'^{\mu\nu}$, we have $A'^{t\nu} = A'^{\nu t} = 0$, $A'^{\nu z} = A'^{z\nu} = 0$, and $A'^{xx} + A'^{yy} = 0$. Now, the $+$ polarization is

$$A_+ = A'^{xx} = \frac{b-c}{2}, \tag{16}$$

while the $\times$ polarization is

$$A_\times = A'^{xy} = 0. \tag{17}$$

(e) We are asked to set all $A^{t\nu}$ and $A^{z\nu}$ elements to zero and subtract half the trace of the remaining matrix. That half trace is equal to $(b+c)/2$. Then, the claim is that $A'^{\mu\nu}$ is equal to

$$A'^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b-\frac{b+c}{2} & 0 & 0 \\ 0 & 0 & c-\frac{b+c}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b-c & 0 & 0 \\ 0 & 0 & c-b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

which is indeed true.

Question 2 (4 points).

Moore Problem 32.3

Solutions:

We have $h^{\text{TT}}_{\mu\nu} = A_{\mu\nu}\cos(k_\alpha x^\alpha)$, where $k_\alpha = (\omega, 0, 0, -\omega)$ and

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

(a) The stress-energy tensor for gravitational waves is given by

$$T^{\text{GW}}_{\mu\nu} = \frac{1}{32\pi G}\left\langle (\partial_\mu h^{\text{TT}}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{\text{TT}}) \right\rangle. \tag{20}$$

Since $h^{\rm TT}_{\mu\nu}$ is only a function of t and z, then only $T^{\rm GW}_{tt}$, $T^{\rm GW}_{zz}$, $T^{\rm GW}_{tz}$, and $T^{\rm GW}_{zt}$ are nonzero. All of these components will involve the contraction $A_{\mu\nu}A^{\mu\nu} = \mathrm{Tr}(A^2)$, which is

$$\begin{aligned}
A_{\mu\nu}A^{\mu\nu} &= \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\
&= \mathrm{Tr}\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+^2 + A_\times^2 & 0 & 0 \\ 0 & 0 & A_+^2 + A_\times^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\
&= 2(A_+^2 + A_\times^2). \qquad (21)
\end{aligned}$$

Thus, the stress-energy tensor components are

$$\begin{aligned}
T^{\rm GW}_{\mu\nu} &= \frac{1}{32\pi G}\Big\langle(\partial_\mu h^{\rm TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{\rm TT})\Big\rangle \\
&= \frac{A_{\rho\sigma}A^{\rho\sigma}}{32\pi G}k_\mu k_\nu\Big\langle \sin^2(k_\alpha x^\alpha)\Big\rangle \\
&= \frac{(A_+^2 + A_\times^2)}{16\pi G}k_\mu k_\nu\Big\langle \sin^2(k_\alpha x^\alpha)\Big\rangle \\
&= \frac{(A_+^2 + A_\times^2)}{32\pi G}k_\mu k_\nu, \qquad (22)
\end{aligned}$$

since averaging the $\sin^2$ term over several wavelengths gives $\langle \sin^2(k_\alpha x^\alpha)\rangle = 1/2$. We thus get

$$T^{\rm GW}_{tt} = \frac{\omega^2(A_+^2 + A_\times^2)}{32\pi G} = T^{\rm GW}_{zz} = -T^{\rm GW}_{tz} = -T^{\rm GW}_{zt}, \qquad (23)$$

since $k_\alpha = (\omega, 0, 0, -\omega)$. In matrix form,

$$T^{\rm GW}_{\mu\nu} = \frac{\omega^2(A_+^2 + A_\times^2)}{32\pi G}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \qquad (24)$$

(b) By definition of the stress-energy tensor, the flow of energy in the i spatial direction is T^{ti} (with the raised indices). Since $T^{\rm GW}_{tx} = T^{\rm GW}_{ty} = 0$, the only nonzero flux of energy is in the z direction. For this, we have

$$\text{flux} = T^{tz}_{\rm GW} = \eta^{tt}\eta^{zz}T^{\rm GW}_{tz} = -T^{\rm GW}_{tz} = \frac{\omega^2(A_+^2 + A_\times^2)}{32\pi G}, \qquad (25)$$

which indeed a flow of energy in the $+z$ direction.

Question 3 (6 points).

Consider two inspiraling black holes with mass $10M_\odot$, where $M_\odot$ is the mass of the sun. Assume the system is located at the centre of our galaxy; let's call this distance to the black holes $r_{\rm gal}$.

Assume that the initial separation is $100\,r_s$, where r_s is the Schwarzschild radius. In the weak field approximation, compute the gravitational wave amplitude $h(t)$ at the LIGO site as a function of time, making use of the quadrupole radiation formula. Assume that we are seeing the system face on. Then, using the formula for the radiated power derived in class, compute the gradual decay of the orbital radius $R(t)$ (using Newtonian physics to relate the energy density radiated to the change in the orbital radius). The approximations cease to be valid once $R(t)$ approaches r_s, so stop the calculation before that point.

Solutions:
We can consider the two inspiring black holes to be point masses orbiting each other. Let's take the orbital plane to be the xy-plane. The positions of the two black holes are then

$$\vec{r}_a = (R(t)\cos\Omega t, R(t)\sin\Omega t, 0), \tag{26}$$

$$\vec{r}_b = (-R(t)\cos\Omega t, -R(t)\sin\Omega t, 0), \tag{27}$$

where the labels a, b denote the two black holes, Ω is the angular frequency, and $R(t)$ is the orbital radius. The total mass density of two $10M_\odot$ black holes is then

$$\rho(t,\boldsymbol{x}) = 10M_\odot\delta(z)\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t) + \delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right). \tag{28}$$

The trace-reversed metric perturbation far away from the inspiring black holes is given by the quadrupole formula

$$H^{ij}(t,\mathbf{r}) = \frac{2G}{r_{\rm gal}}\ddot{\mathcal{I}}^{ij}(t-r), \tag{29}$$

where

$$\mathcal{I}^{ij}(t) = \int (x'^i x'^j - \frac{1}{3}\eta^{ij}r'^2)\rho(t,\boldsymbol{x}')d^3x', \tag{30}$$

is the reduced quadrupole moment of the two black holes. The first part of the calculation is to evaluate this quadrupole moment. Let me first compute the second term in Eq. (30), since this term will be the same for all diagonal elements (and zero for off-diagonal elements).

$$\begin{aligned}
-\frac{1}{3}\eta^{ij}\int(r^2)\rho(t,\boldsymbol{x})d^3x &= -\frac{10M_\odot}{3}\delta^{ij}\int(x^2+y^2+z^2)\delta(z)\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t)\right.\\
&\qquad\left.+\delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right)dxdydz\\
&= -\frac{10M_\odot}{3}\delta^{ij}\int(x^2+y^2)\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t)\right.\\
&\qquad\left.+\delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right)dxdy\\
&= -\frac{20M_\odot}{3}\delta^{ij}\left(R^2(t)\cos^2\Omega t + R^2(t)\sin^2\Omega t\right)\\
&= -\frac{20R^2(t)M_\odot}{3}\delta^{ij}.
\end{aligned} \tag{31}$$

Now, the first term in Eq. (30)

$$\begin{aligned}
\int(x^ix^j)\rho(t,\boldsymbol{x})dxdydz = 10M_\odot\int(x^ix^j)\delta(z)\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t)\right.\\
\left.+\delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right)dxdydz.
\end{aligned} \tag{32}$$

Clearly, if either i or $j = 3$, this is clearly 0 since the $\delta(z)$ delta function will always set $z = 0$. The other cases are

$$\int (x^2)\rho(t,\boldsymbol{x})dxdydz = 20R^2(t)M_\odot \cos^2\Omega t = 10R^2(t)M_\odot\left(1+\cos 2\Omega t\right), \tag{33}$$

$$\int (xy)\rho(t,\boldsymbol{x})dxdydz = 20M_\odot R^2(t)\cos\Omega t\sin\Omega t = 10R^2(t)M_\odot \sin 2\Omega t, \tag{34}$$

$$\int (y^2)\rho(t,\boldsymbol{x})dxdydz = 20M_\odot R^2(t)\sin^2\Omega t = 10R^2(t)M_\odot\left(1-\cos 2\Omega t\right). \tag{35}$$

Putting everything together, we get

$$\mathcal{I}^{ij}(t) = \frac{10R^2(t)M_\odot}{3}\begin{pmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{36}$$

Now, let's use some classical mechanics to relate Ω to R and the black hole masses. Basically, we want to equal the gravitational force that one black hole feels to the "centrifugal " force

$$\frac{GM^2}{(2R(t))^2} = M\Omega^2 R(t), \qquad \Rightarrow \Omega = \left(\frac{GM}{4R^3(t)}\right)^{1/2}, \tag{37}$$

where $M = 10M_\odot$. Now, let's use the quadrupole formula to compute the trace-reversed metric perturbation far away. In doing so, we will assume that the rate of change of the orbital radius $R(t)$ is much smaller than Ω (we will check later that this is indeed a good approximation). We have

$$\begin{aligned} H^{ij}(t,\mathbf{r}) &= \frac{2G(10M_\odot)R^2(t_r)(2\Omega)^2}{r_{\text{gal}}}\begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{200G^2M_\odot^2}{r_{\text{gal}}R(t_r)}\begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{38}$$

where $t_r \equiv t - r$. Now, if the system appears to us face on, we are basically looking at the system down the z-axis. So, we are detecting gravitational waves traveling down the z-axis, i.e., with a wave vector pointing in the z-direction. For such waves, the trace-reversed metric perturbation above is equal to the transverse-traceless perturbation $H^{ij} = H^{ij}_{\text{TT}} = h^{ij}_{\text{TT}}$ (since $k_\mu H^{\mu\nu} = 0$ here). Thus, the LIGO detector will see the wave

$$h^{ij}_{\text{TT}}(t) = \frac{200G^2M_\odot^2}{r_{\text{gal}}R(t_r)}\begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

This is a wave with amplitude

$$A_+ = A_\times = \frac{200G^2M_\odot^2}{r_{\text{gal}}R(t_r)}, \tag{40}$$

but where the two polarization are 90 degrees out of phase (this lead to a *circularly* polarized wave). We can also estimate the initial signal strength since the separation of the black holes is $100r_s$ at first (i.e. $R = 50r_s$). Since $r_s = 2GM = 20GM_\odot$, we have

$$A_+ = A_\times = \frac{GM_\odot}{5r_{\rm gal}}, \tag{41}$$

initially. Since $r_{\rm gal} \sim 8$ kpc, we have $A_+ = A_\times \sim 10^{-18}$, which is a signal that LIGO can easily detect. Now, let's turn our attention to the orbital decay caused by the energy taken away by the gravitational waves. The total power emitted in gravitational waves by the inspiring black holes is

$$P = \frac{G}{5}\left\langle \dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}^{jk}\right\rangle. \tag{42}$$

Using Eq. (36) above, we have

$$\dddot{\mathcal{I}}^{ij}(t) = 10R^2(t)M_\odot(2\Omega)^3\begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{43}$$

where again we have assumed that $R(t)$ changes very slowly. Then,

$$\begin{aligned}\left\langle \dddot{\mathcal{I}}^{ij}\dddot{\mathcal{I}}_{ij}\right\rangle &= 100M_\odot^2R^4(t)(2\Omega)^6\left\langle \sin^2 2\Omega t + \sin^2 2\Omega t + \cos^2 2\Omega t + \cos^2 2\Omega t\right\rangle \\ &= 12800M_\odot^2R^4(t)\Omega^6 \\ &= 2\times 10^5\frac{G^3M_\odot^5}{R^5(t)},\end{aligned} \tag{44}$$

and the total power emitted is

$$P = 2\times 10^5\frac{G^4M_\odot^5}{5R^5(t)}. \tag{45}$$

Using Newtonian physics, the total energy of the inspiring black holes is given by

$$E = \frac{1}{2}M_aR^2(t)\Omega^2 + \frac{1}{2}M_bR^2(t)\Omega^2 - \frac{GM_aM_b}{2R(t)}, \tag{46}$$

where $M_a = M_b = 10M_\odot$. Substituting Ω from above, this reduces to

$$E = -\frac{G(10M_\odot)^2}{4R(t)}. \tag{47}$$

The rate of change of the energy is then

$$\frac{dE}{dt} = \frac{G(10M_\odot)^2}{4R^2(t)}\frac{dR(t)}{dt}. \tag{48}$$

This must be equal to the power emitted in gravitational waves

$$\frac{G(10M_\odot)^2}{4R^2(t)}\frac{dR(t)}{dt} = -\frac{2G^4(10M_\odot)^5}{5R^5(t)} \qquad \Rightarrow \qquad R^3(t)\frac{dR(t)}{dt} = -\frac{8G^3(10M_\odot)^3}{5}. \tag{49}$$

Integrating on both sides yields

$$\begin{aligned}\int_{50r_s}^{R} R'^3 dR' &= -\frac{8G^3(10M_\odot)^3}{5}\int_0^t dt' \\ \frac{R(t)^4-(50r_s)^4}{4} &= -\frac{8G^3(10M_\odot)^3}{5}t \\ R(t) &= \left((50r_s)^4 - \frac{32G^3(10M_\odot)^3}{5}t\right)^{1/4} \\ R(t) &= (10GM_\odot)^{3/4}\left(10^9 GM_\odot - \frac{32}{5}t\right)^{1/4}. \end{aligned} \tag{50}$$

Finally, let's check that the rate of change of the orbital radius is indeed much slower than the angular velocity at the beginning of the in-spiral. Basically, we want to show that

$$\left|\frac{1}{R(t)}\frac{dR(t)}{dt}\right|_{t\sim 0} \ll \frac{\Omega}{2\pi}. \tag{51}$$

We have

$$\left|\frac{1}{R(t)}\frac{dR(t)}{dt}\right|_{t\sim 0} = \frac{8G^3M^3}{5R^4(0)} = \frac{32\pi}{5}\left(\frac{GM}{R(0)}\right)^{5/2}\frac{\Omega}{2\pi} = \frac{32\pi}{5}\left(\frac{1}{100}\right)^{5/2}\frac{\Omega}{2\pi} \simeq 2\times 10^{-4}\frac{\Omega}{2\pi}, \tag{52}$$

and indeed the rate of change of the orbital radius is very small compare to the orbital frequency.