Physics 480/581
General Relativity

Homework Assignment 14 Solutions

Question 1 (3 points).

Our current universe appears to be dominated by a cosmological constant. Compute the age of our universe assuming that today (when the Hubble expansion rate is $H_0 \simeq 70$ km/s/Mpc) 70% of the energy is in the form of the cosmological constant and 30% is in the form of cold matter.
Solutions:
The age of the Universe is given by

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_0^1 \frac{da}{aH}, \end{aligned} \tag{1}$$

where the Hubble rate is given by

$$H = H_0 \left(\Omega_{\rm m} a^{-3} + \Omega_\Lambda\right)^{1/2}. \tag{2}$$

We thus have

$$t_0 = \frac{1}{H_0\sqrt{\Omega_{\rm m}}} \int_0^1 \frac{\sqrt{a}da}{\left(1 + \frac{\Omega_\Lambda}{\Omega_{\rm m}} a^3\right)^{1/2}}. \tag{3}$$

Take $u^2 = (\Omega_\Lambda/\Omega_{\rm m})a^3$, $\sqrt{a}da = (2/3)(\Omega_\Lambda/\Omega_{\rm m})^{1/2} du$,

$$\begin{aligned} t_0 &= \frac{2}{3H_0\sqrt{\Omega_{\rm m}}\sqrt{(\Omega_\Lambda/\Omega_{\rm m})}} \int_0^{\sqrt{(\Omega_\Lambda/\Omega_{\rm m})}} \frac{du}{(1+u^2)^{1/2}} \\ &= \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1}\sqrt{\frac{\Omega_\Lambda}{\Omega_{\rm m}}}. \end{aligned} \tag{4}$$

Using $\Omega_\Lambda = 0.7$, $\Omega_{\rm m} = 0.3$, and $H_0 = 1.023h \times 10^{-10}$ year^{-1}, where $h \equiv H_0/(100\,\text{km/s/Mpc}) = 0.7$, we obtain

$$t_0 \simeq 13.5 \times 10^9 \text{years}. \tag{5}$$

Question 2 (4 points).

Let's consider the flat FLRW metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2], \tag{6}$$

where $a(t)$ is the scale factor. For this problem, use the value of the cosmological parameters provided in the inner front cover of Moore and assume a realistic universe filled with matter, radiation, and a cosmological constant. In cosmology, we often referred to epochs in the evolution of the Universe in terms of their *redshift*, which is related to the scale factor by $a(t) = 1/(1+z)$.

(a) Using the fact that photons always travel on null trajectories ($ds^2 = 0$), compute the total comoving distance that a photon will travel from the Big Bang at $t = 0$ to the epoch of recombination at redshift $z = 1090$.

Solutions:

Assuming the photon is traveling in the radial direction r, the condition $ds^2 = 0$ reduces to $dr = dt/a(t)$. Integrating this equation yields the comoving distance traveled by the photon

$$\begin{aligned} d_{\rm c} &= \int dr \\ &= \int \frac{dt}{a(t)} \\ &= \int_0^{a_{\rm rec}} \frac{da}{a^2 H}. \end{aligned} \tag{7}$$

Here, the Hubble rate is

$$H = H_0 \left(\Omega_{\rm r} a^{-4} + \Omega_{\rm m} a^{-3} + \Omega_\Lambda\right)^{1/2}. \tag{8}$$

Note that since we are only integrating over small values of the scale factor from 0 to $a_{\rm rec} = 1/1091$, the two first terms in the above equation will always by far dominate over the cosmological constant term. So, let us neglect dark energy here and only consider matter and radiation. The comoving distance is then

$$\begin{aligned} d_{\rm c}(0, t_{\rm rec}) &= \frac{1}{H_0} \int_0^{a_{\rm rec}} \frac{da}{a^2 \left(\Omega_{\rm r} a^{-4} + \Omega_{\rm m} a^{-3}\right)^{1/2}} \\ &= \frac{1}{H_0} \int_0^{a_{\rm rec}} \frac{da}{\left(\Omega_{\rm r} + \Omega_{\rm m} a\right)^{1/2}} \\ &= \frac{2}{H_0 \Omega_{\rm m}} \left(\Omega_{\rm r} + \Omega_{\rm m} a\right)^{1/2} \Big|_0^{a_{\rm rec}} \\ &= \frac{2}{H_0 \Omega_{\rm m}} \left[\left(\Omega_{\rm r} + \Omega_{\rm m} a_{\rm rec}\right)^{1/2} - \Omega_{\rm r}^{1/2}\right]. \end{aligned} \tag{9}$$

Using the values $\Omega_{\rm r} = 0.000084$, $\Omega_{\rm m} = 0.272$, and $\Omega_\Lambda = 0.728$ found on the front inner cover of Moore, I get

$$d_{\rm c}(0, t_{\rm rec}) = \frac{6.68 \times 10^{-2}}{H_0} \approx 2.85 \times 10^5\,\text{kpc}, \tag{10}$$

where we used $H_0 = 7.61 \times 10^{-27}$ m^{-1}, and the fact that $1\,\text{pc} = 3.0857 \times 10^{16}$ m.

(b) Now compute the total comoving distance that a photon will travel from the epoch of recombination to the present time ($z = 0$).

Solutions:

This time, we cannot neglect the presence of the cosmological constant in the Hubble rate, so the distance traveled by the photon from $a = a_{\rm rec}$ to $a = 1$ is

$$\begin{aligned} d_c(t_{\rm rec}, t_0) &= \frac{1}{H_0} \int_{a_{\rm rec}}^{1} \frac{da}{a^2 \left(\Omega_{\rm r} a^{-4} + \Omega_{\rm m} a^{-3} + \Omega_\Lambda\right)^{1/2}} \\ &= \frac{1}{H_0} \int_{a_{\rm rec}}^{1} \frac{da}{\left(\Omega_{\rm r} + \Omega_{\rm m} a + \Omega_\Lambda a^4\right)^{1/2}}. \end{aligned} \tag{11}$$

This integral is easiest done numerically using the values $\Omega_{\rm r} = 0.000084$, $\Omega_{\rm m} = 0.272$, and $\Omega_\Lambda = 0.728$ found on the front inner cover of Moore. With these I find

$$\int_{a_{\rm rec}}^{1} \frac{da}{\left(\Omega_{\rm r} + \Omega_{\rm m} a + \Omega_\Lambda a^4\right)^{1/2}} \simeq 3.305. \tag{12}$$

The comoving distance traveled by the photon from $a = a_{\rm rec}$ to $a = 1$ is

$$d_c(t_{\rm rec}, t_0) = \frac{3.305}{H_0} \approx 1.41 \times 10^7 \,\text{kpc}. \tag{13}$$