PHYS 480/581
General Relativity

Homework Assignment 2

Due date: Wednesday 01/31/2024 5pm, submitted electronically on UNM Canvas

Question 1 (2 points).

Particle physicists are so used to setting $c = 1$ that they measure mass in units of energy. For instance, they use electron volts (1 eV $= 1.6 \times 10^{-12}\,\mathrm{erg} = 1.8 \times 10^{-33}\,\mathrm{g}$), or more commonly, keV, MeV, and GeV ($10^3\,\mathrm{eV}$, $10^6\,\mathrm{eV}$, and $10^9\,\mathrm{eV}$, respectively). The muon has been measured to have a rest mass of $106\,\mathrm{MeV}$ and a rest frame lifetime of 2.19×10^{-6} seconds. Imagine that a muon is moving in the circular storage ring of a particle accelerator, 1 kilometer in diameter, such that the muon's total energy is $1000\,\mathrm{GeV}$. How long would it appear to live from the experimenter's point of view? How many radians would it travel around the ring?

Question 2 (4 points).

Moore Problem 3.1

Question 3 (4 points).

The principle of relativity states that the laws of physics are the same in every inertial reference frames. Quantitatively, one thing we mean by this is that all inertial observers will agree on the norm of four-vectors. The Lorentz transformations are actually *defined* as the set of linear transformations between inertial frames that leave the norm of four-vectors invariant.

Suppose we have a four-vector $\boldsymbol{p}$ in some inertial frame S. A different observer in an inertial frame S' will see the four-vector $\boldsymbol{p}' = \boldsymbol{\Lambda}\boldsymbol{p}$, where $\boldsymbol{\Lambda}$ is a Lorentz transformation matrix.

(a) Using the fact that $\boldsymbol{p}^2 = \boldsymbol{p}'^2$, show that the Lorentz transformation matrices $\boldsymbol{\Lambda}$ obey the following identity

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}, \tag{1}$$

where $\boldsymbol{\eta}$ is the Minkowski metric, and "T" denotes matrix transposition. If you prefer, you can do this computation in component notation, in which case the result is

$$\eta_{\rho\sigma} = \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma}. \tag{2}$$

Matrices $\boldsymbol{\Lambda}$ satisfying Eq. (1) form a group under matrix multiplication called $\mathrm{O}(1,3)$ (where O stands for orthogonal since Eq. (1) is essentially the orthogonality condition for matrices M, i.e. $M^{\mathrm{T}}M = MM^{\mathrm{T}} = \mathbb{1}$, but with respect to the Minkowski metric).

(b) Use the properties of the matrix determinant to show that

$$\det \boldsymbol{\Lambda} = \pm 1. \tag{3}$$

Here, transformations with $\det \boldsymbol{\Lambda} = 1$ correspond to spacetime rotations (i.e. regular 3D rotation and Lorentz boosts), while those with $\det \boldsymbol{\Lambda} = -1$ correspond to reflections (or parity transformations), which essentially turn a right-handed reference frame into a left-handed one. In general, we prefer our Lorentz transformations to preserve the handedness of our reference frames and we always work with $\det \boldsymbol{\Lambda} = 1$. With this choice and Eq. (1) above, the matrices $\boldsymbol{\Lambda}$ form a group under matrix multiplication called $\mathrm{SO}(1,3)$ (where S stands for "special").

(c) Even with $\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}$ and $\det \boldsymbol{\Lambda} = 1$, this is not exactly the kind of Lorentz transformations we want in physics. In particular, we would like all our Lorentz transformations to be smoothly connected to the identity, that is, for some $\epsilon \ll 1$,

$$\boldsymbol{\Lambda} \simeq \mathbb{1} + \epsilon \mathbf{X}, \tag{4}$$

where $\mathbf{X}$ is a matrix (called a group *generator*) that has the required properties to ensure that $\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}$ and $\det \boldsymbol{\Lambda} = 1$. Note that Eq. (4) is a very reasonable request: if two inertial frames are barely moving with respect to each other, then these two frames are nearly the same and the Lorentz transformation between them should be nearly the identity. Now, consider the matrix

$$\boldsymbol{\Lambda}_{\mathrm{tp}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{5}$$

which is a combination of time reversal and parity transformations. Show that this matrix indeed satisfies $\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}$ and $\det \boldsymbol{\Lambda} = 1$, but then argue that this matrix is **not** smoothly connected to the identity matrix. Clearly, we want to exclude this possibility from our space of Lorentz transformations.

Using Eq. (2) above, show that the component $\Lambda^0{}_0$ of matrices satisfying $\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}$ can have values falling in two *distinct* intervals on the real axis. Identify which choice of interval effectively eliminates $\boldsymbol{\Lambda}_{\mathrm{tp}}$ from our set of Lorentz transformations, and then summarize the three key properties that matrices $\boldsymbol{\Lambda}$ must satisfy to be physical Lorentz transformations. Such matrices are said to form the proper orthochronous Lorentz group.