Physics 480/581
General Relativity

Homework Assignment 2 Solutions

Question 1 (2 points).

Particle physicists are so used to setting $c = 1$ that they measure mass in units of energy. For instance, they use electron volts (1 eV $= 1.6 \times 10^{-12}$ erg $= 1.8 \times 10^{-33}$ g), or more commonly, keV, MeV, and GeV (10^3 eV, 10^6 eV, and 10^9 eV, respectively). The muon has been measured to have a rest mass of 106 MeV and a rest frame lifetime of 2.19×10^{-6} seconds. Imagine that a muon is moving in the circular storage ring of a particle accelerator, 1 kilometer in diameter, such that the muon's total energy is 1000 GeV. How long would it appear to live from the experimenter's point of view? How many radians would it travel around the ring?
Solutions:

The first step is to find the speed of a $E = 1000\,\mathrm{GeV}$ muon in the lab frame. In relativity, the speed is given by $v = p/E$, where p is the (relativistic) three-momentum and E is the energy. Using $E^2 = m^2 + p^2$, the speed is

$$v = \frac{\sqrt{E^2 - m^2}}{E} = 0.99999999438 \approx 1, \tag{1}$$

so the muon is basically propagating at the speed of light. The relativistic γ factor is

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E}{m}, \tag{2}$$

where we used Eq. (1). Now, intuitively, the lifetime of the muon in the lab frame will be the rest frame lifetime times the dilation factor γ (this is the right answer). But let's say you didn't remember that. Instead, let's relate the proper time of the muon to the coordinate time of an observer in the lab frame. The proper time is of course defined by $d\tau = \sqrt{-ds^2}$. Since the muon is going around a circular storage ring, it is useful to use polar coordinates (r, θ) to describe the spatial part of the line element ds^2. With this choice, the muon proper time elapsed within some coordinate time t is

$$\tau = \int \sqrt{-ds^2} = \int \sqrt{dt^2 - dr^2 - r^2 d\theta^2} \tag{3}$$

$$= \int_0^t dt \sqrt{1 - \left(\frac{dr}{dt}\right)^2 - r^2 \left(\frac{d\theta}{dt}\right)^2}. \tag{4}$$

Now, the muon is moving around the ring so $dr/dt = 0$ since $r = R = 0.5$ km is fixed, and $r d\theta/dt = R d\theta/dt = v$, the speed computed above. Thus, the muon proper time is

$$\tau = \int_0^t dt \sqrt{1 - v^2} = \left(\sqrt{1 - v^2}\right) t, \tag{5}$$

and thus

$$t = \frac{\tau}{\sqrt{1 - v^2}} = \gamma \tau, \tag{6}$$

as per our intuition. Note that this answer is independent of the fact that the muon is going around a ring (that is, we would have gotten the same answer is the muon was going in a straight line with velocity v). Of course, the proper time here is the time measured by a clock traveling with the muon (i.e in its rest frame). Setting this proper time equal to the muon lifetime τ_μ, the observer in the lab frame will observe this lifetime to be

$$t = \gamma \tau_\mu = \frac{E}{m} \tau_\mu \approx 0.0207\,\mathrm{s}. \tag{7}$$

In this time interval, the muon travels a distance

$$L = vt = \frac{\sqrt{E^2 - m^2}}{m} \tau_\mu = R\Delta\theta, \tag{8}$$

where $\Delta\theta$ is the number of radians that the muon travel around the ring. We then get

$$\Delta\theta = \frac{\sqrt{E^2 - m^2}}{mR} \tau_\mu \approx \frac{E\tau_\mu}{mR} = 1.24 \times 10^4 \mathrm{rad}. \tag{9}$$

This means that the muon will go around the ring nearly two thousand times before decaying.

Question 2 (4 points).

Moore Problem 3.1
Solutions:

(a) We simply have

$$u^x = \frac{dx}{d\tau} = \sinh(g\tau). \tag{10}$$

(b)

$$\boldsymbol{u} \cdot \boldsymbol{u} = -(u^t)^2 + (u^x)^2 = -1 \quad \rightarrow \quad u^t = \sqrt{1 + (u^x)^2} = \cosh(g\tau) \tag{11}$$

(c) The speed of the object is

$$v = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{u^x}{u^t} = \frac{\sinh(g\tau)}{\cosh(g\tau)} = \tanh(g\tau), \tag{12}$$

which is always than 1 since $|\tanh x| < 1$ for all x.

(d) Since $u^t = dt/d\tau = \cosh(g\tau)$, we have that

$$t = \int_0^\tau \cosh(g\tau')d\tau' = \frac{1}{g}\sinh(g\tau). \tag{13}$$

Thus, $gt = \sinh(g\tau)$.

(e) Putting everything together, we have

$$u^x = gt, \qquad u^t = \sqrt{1 + (gt)^2}, \qquad v = \frac{gt}{\sqrt{1 + (gt)^2}}. \tag{14}$$

Question 3 (4 points).

The principle of relativity states that the laws of physics are the same in every inertial reference frames. Quantitatively, one thing we mean by this is that all inertial observers will agree on the norm of four-vectors. The Lorentz transformations are actually *defined* as the set of linear transformations between inertial frames that leave the norm of four-vectors invariant.

Suppose we have a four-vector $\boldsymbol{p}$ in some inertial frame S. A different observer in an inertial frame S' will see the four-vector $\boldsymbol{p}' = \boldsymbol{\Lambda}\boldsymbol{p}$, where $\boldsymbol{\Lambda}$ is a Lorentz transformation matrix.

(a) Using the fact that $\boldsymbol{p}^2 = \boldsymbol{p}'^2$, show that the Lorentz transformation matrices $\boldsymbol{\Lambda}$ obey the following identity

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}, \tag{15}$$

where $\boldsymbol{\eta}$ is the Minkowski metric, and "T" denotes matrix transposition. If you prefer, you can do this computation in component notation, in which case the result is

$$\eta_{\rho\sigma} = \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma}. \tag{16}$$

Matrices $\boldsymbol{\Lambda}$ satisfying Eq. (15) form a group under matrix multiplication called $\mathrm{O}(1,3)$ (where O stands for orthogonal since Eq. (15) is essentially the orthogonality condition for matrices M, i.e. $M^{\mathrm{T}}M = MM^{\mathrm{T}} = \mathbb{1}$, but with respect to the Minkowski metric).

Solutions: Let us first use matrix notation. The first thing to realize is that $\boldsymbol{p}^2 = \boldsymbol{p}\cdot\boldsymbol{p} = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{p}$, by the definition of the inner product. We then have

$$\begin{aligned}
\boldsymbol{p}^2 &= \boldsymbol{p}'^2 \\
\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{p} &= \boldsymbol{p}'^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{p}' \\
\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{p} &= (\boldsymbol{\Lambda}\boldsymbol{p})^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}\boldsymbol{p} \\
\boldsymbol{p}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{p} &= \boldsymbol{p}^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}\boldsymbol{p}.
\end{aligned} \tag{17}$$

Since this is true for an arbitrary vector $\boldsymbol{p}$, we then have

$$\boldsymbol{\eta} = \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\boldsymbol{\Lambda}. \tag{18}$$

If you prefer to do it in component notation, we have

$$\begin{aligned}
\boldsymbol{p}^2 &= \boldsymbol{p}'^2 \\
\eta_{\rho\sigma}p^{\rho}p^{\sigma} &= \eta_{\mu\nu}p'^{\mu}p'^{\nu} \\
\eta_{\rho\sigma}p^{\rho}p^{\sigma} &= \eta_{\mu\nu}(\Lambda^{\mu}{}_{\rho}p^{\rho})(\Lambda^{\nu}{}_{\sigma}p^{\sigma}) \\
\eta_{\rho\sigma}p^{\rho}p^{\sigma} &= \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma}p^{\rho}p^{\sigma} \\
(\eta_{\rho\sigma} - \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma})p^{\rho}p^{\sigma} &= 0,
\end{aligned} \tag{19}$$

which for arbitrary vector $\boldsymbol{p}$ implies

$$\eta_{\rho\sigma} = \Lambda^{\mu}{}_{\rho}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma}. \tag{20}$$

Note that if we want to convert from component to matrix notation, we need to place summed-over indices next to each other. This requires changing the order of μ and ρ in the first $\mathbf{\Lambda}$ matrix, which corresponds to transposing the matrix, i.e.

$$\eta_{\rho\sigma} = (\Lambda^{\mathrm{T}})_{\rho}{}^{\mu}\eta_{\mu\nu}\Lambda^{\nu}{}_{\sigma}, \tag{21}$$

which in matrix notation is

$$\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}. \tag{22}$$

(b) Use the properties of the matrix determinant to show that

$$\det \mathbf{\Lambda} = \pm 1. \tag{23}$$

Here, transformations with $\det \mathbf{\Lambda} = 1$ correspond to spacetime rotations (i.e. regular 3D rotation and Lorentz boosts), while those with $\det \mathbf{\Lambda} = -1$ correspond to reflections (or parity transformations), which essentially turn a right-handed reference frame into a left-handed one. In general, we prefer our Lorentz transformations to preserve the handedness of our reference frames and we always work with $\det \mathbf{\Lambda} = 1$. With this choice and Eq. (15) above, the matrices $\mathbf{\Lambda}$ form a group under matrix multiplication called SO(1, 3) (where S stands for "special").

Solutions:

We simply use the fact that the determinant of a matrix product is the product of the determinant, that is,

$$\begin{aligned}
\det \boldsymbol{\eta} &= \det(\mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}) \\
\det \boldsymbol{\eta} &= (\det \mathbf{\Lambda}^{\mathrm{T}})(\det \boldsymbol{\eta})(\det \mathbf{\Lambda}) \\
1 &= (\det \mathbf{\Lambda}^{\mathrm{T}})(\det \mathbf{\Lambda}) \\
1 &= (\det \mathbf{\Lambda})^2,
\end{aligned} \tag{24}$$

where we have use the fact that $\det \boldsymbol{\eta} = -1 \neq 0$, and the fact that $\det \mathbf{\Lambda}^{\mathrm{T}} = \det \mathbf{\Lambda}$. The above immediately implies that

$$\det \mathbf{\Lambda} = \pm 1. \tag{25}$$

since $\mathbf{\Lambda}$ are real matrices. As mentioned above, we only keep $\mathbf{\Lambda}$ matrices with $\det \mathbf{\Lambda} = 1$ as part of the Lorentz group.

(c) Even with $\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}$ and $\det \mathbf{\Lambda} = 1$, this is not exactly the kind of Lorentz transformations we want in physics. In particular, we would like all our Lorentz transformations to be smoothly connected to the identity, that is, for some $\epsilon \ll 1$,

$$\mathbf{\Lambda} \simeq \mathbb{1} + \epsilon \mathbf{X}, \tag{26}$$

where $\mathbf{X}$ is a matrix (called a group *generator*) that has the required properties to ensure that $\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}$ and $\det \mathbf{\Lambda} = 1$. Note that Eq. (26) is a very reasonable request: if two inertial frames are barely moving with respect to each other, then these two frames are nearly the same and the Lorentz transformation between them should be nearly the identity. Now, consider the matrix

$$\mathbf{\Lambda}_{\mathrm{tp}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{27}$$

which is a combination of time reversal and parity transformations. Show that this matrix indeed satisfies $\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}$ and $\det\mathbf{\Lambda} = 1$, but then argue that this matrix is **not** smoothly connected to the identity matrix. Clearly, we want to exclude this possibility from our space of Lorentz transformations.

Using Eq. (16) above, show that the component $\Lambda^0{}_0$ of matrices satisfying $\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}$ can have values falling in two *distinct* intervals on the real axis. Identify which choice of interval effectively eliminates $\mathbf{\Lambda}_{\mathrm{tp}}$ from our set of Lorentz transformations, and then summarize the three key properties that matrices $\mathbf{\Lambda}$ must satisfy to be physical Lorentz transformations. Such matrices are said to form the proper orthochronous Lorentz group.

Solutions:

The important thing to notice is that $\mathbf{\Lambda}_{\mathrm{tp}} = -\mathbb{1}$, from which we immediately get

$$\begin{aligned}\mathbf{\Lambda}_{\mathrm{tp}}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}_{\mathrm{tp}} &= (-\mathbb{1})^{\mathrm{T}}\boldsymbol{\eta}(-\mathbb{1}) \\ &= \mathbb{1}\boldsymbol{\eta}\mathbb{1} \\ &= \boldsymbol{\eta},\end{aligned} \tag{28}$$

so indeed $\mathbf{\Lambda}_{\mathrm{tp}}$ satisfies the orthogonality condition. Similarly

$$\begin{aligned}\det\mathbf{\Lambda}_{\mathrm{tp}} &= \det(-\mathbb{1}) \\ &= (-1)^4\det\mathbb{1} \\ &= 1,\end{aligned} \tag{29}$$

where we have used the property of the determinant that $\det c\mathbf{A} = c^n\det\mathbf{A}$, where n is the size of the square matrix $\mathbf{A}$ and c is a real number. Thus, this condition is also satisfied. Now, since $\mathbf{\Lambda}_{\mathrm{tp}} = -\mathbb{1}$, it is impossible to find an $\epsilon \ll 1$ such that

$$\mathbf{\Lambda}_{\mathrm{tp}} = -\mathbb{1} \simeq \mathbb{1} + \epsilon\mathbf{X} \qquad \text{(impossible if } \epsilon \ll 1\text{)}, \tag{30}$$

where it is understood that every entry in the product $\epsilon\mathbf{X} \ll 1$. Essentially, minus the identity matrix is not infinitesimally connected to the identity. Now let's look at the condition on the $0-0$ entry in the $\mathbf{\Lambda}$ matrices. Using component notation and the Einstein summation convention, we have

$$\begin{aligned}\eta_{00} &= \Lambda^\mu{}_0\eta_{\mu\nu}\Lambda^\nu{}_0 \\ -1 &= \Lambda^0{}_0\eta_{00}\Lambda^0{}_0 + \Lambda^i{}_0\eta_{ij}\Lambda^j{}_0 \\ -1 &= -(\Lambda^0{}_0)^2 + \Lambda^i{}_0\delta_{ij}\Lambda^j{}_0 \\ -1 &= -(\Lambda^0{}_0)^2 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2 \\ (\Lambda^0{}_0)^2 &= 1 + \sum_{i=1}^{3}(\Lambda^i{}_0)^2.\end{aligned} \tag{31}$$

Since $\mathbf{\Lambda}$ are real matrices, note that we have

$$\sum_{i=1}^{3}(\Lambda^i{}_0)^2 \geq 0. \tag{32}$$

Thus, we have two distinct possibilities for ${\Lambda^0}_0$,

$$ {\Lambda^0}_0 = \pm\sqrt{1 + \sum_{i=1}^{3} ({\Lambda^i}_0)^2}, \tag{33} $$

which implies that we either have ${\Lambda^0}_0 \geq 1$ or ${\Lambda^0}_0 \leq -1$. Thus, ${\Lambda^0}_0$ can indeed lie within two distinct intervals on the real axis. If we want to eliminate a transformation like $\mathbf{\Lambda}_{\mathrm{tp}}$ from our set of Lorentz transformations, we can then restrict our transformations to have ${\Lambda^0}_0 \geq 1$ always.

In summary, the Lorentz transformations $\mathbf{\Lambda}$ that we consider in physics have the following properties

$$ \{\boldsymbol{\eta} = \mathbf{\Lambda}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{\Lambda}, \det\mathbf{\Lambda} = 1, {\Lambda^0}_0 \geq 1\}. \tag{34} $$