PHYS 480/581
General Relativity

Homework Assignment 3 Solutions

Question 1 (7 points).

Imagine we have a tensor (matrix) $X^{\mu\nu}$ and a vector V^μ, with components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \qquad V^\mu = (-1, 2, 0, -2)\,. \tag{1}$$

Assuming that these two objects live in flat spacetime with a Minkowski metric $\eta_{\mu\nu}$, find the components of:

(a) $X^\mu{}_\nu$
Solutions:

$$X^\mu{}_\nu = X^{\mu\alpha}\eta_{\alpha\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 & -1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} \tag{2}$$

(b) $X_\mu{}^\nu$
Solutions:

$$X_\mu{}^\nu = \eta_{\mu\alpha}X^{\alpha\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & -1 & 1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \tag{3}$$

(c) $X^{(\mu\nu)} \equiv \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right)$
Solutions:

$$X^{(\mu\nu)} = \frac{1}{2}\left(\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 1 & 3 & 0 & 1 \\ -1 & 2 & 0 & -2 \end{pmatrix}\right) = \begin{pmatrix} 2 & -\frac{1}{2} & 0 & -\frac{3}{2} \\ -\frac{1}{2} & 0 & 2 & \frac{3}{2} \\ 0 & 2 & 0 & \frac{1}{2} \\ -\frac{3}{2} & \frac{3}{2} & \frac{1}{2} & -2 \end{pmatrix}. \tag{4}$$

(d) $X_{[\mu\nu]} \equiv \frac{1}{2}\left(X_{\mu\nu} - X_{\nu\mu}\right)$

Solutions: First, let's find $X_{\mu\nu}$

$$X_{\mu\nu} = \eta_{\mu\alpha} X^{\alpha\beta} \eta_{\beta\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

$$= \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix}. \tag{6}$$

Thus,

$$X_{[\mu\nu]} = \frac{1}{2}\left(\begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 3 & 0 & 1 \\ 1 & 2 & 0 & -2 \end{pmatrix} \right) = \begin{pmatrix} 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ 1 & -1 & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \tag{7}$$

(e) $X^{\lambda}{}_{\lambda}$
Solutions:

$$X^{\lambda}{}_{\lambda} = X^{\lambda\alpha}\eta_{\alpha\lambda} = \mathrm{Tr}\left[\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] = -4. \tag{8}$$

(f) $V^{\mu}V_{\mu}$
Solutions:

$$V^{\mu}V_{\mu} = \eta_{\mu\nu}V^{\mu}V^{\nu} = -(V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2 = 7. \tag{9}$$

(g) $V_{\mu}X^{\mu\nu}$
Solutions:

$$V_{\mu}X^{\mu\nu} = V^{\alpha}\eta_{\alpha\mu}X^{\mu\nu} = (-1, 2, 0, -2) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix} \tag{10}$$

$$= (4, -2, 5, 7)\,. \tag{11}$$

Question 2 (2 points).

The electromagnetic Lagrangian density is $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. With the help of Eq. (4.14) in Moore, write down $\mathcal{L}$ in terms of the $\vec{E}$ and $\vec{B}$ field components.
Solutions: First, the components of $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = \eta_{\mu\alpha}F^{\alpha\beta}\eta_{\beta\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

and thus

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{13}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \tag{14}$$

$$= -\frac{1}{4}\left(2F_{0i}F^{0i} + 2\sum_{j>i} F_{ij}F^{ij}\right), \tag{15}$$

where we use the antisymmetric nature of $F^{\mu\nu}$ to realize that $F_{0i}F^{0i} = F_{i0}F^{i0}$. Thus

$$\mathcal{L} = -\frac{1}{2}\left(-E_x^2 - E_y^2 - E_z^2 + B_x^2 + B_y^2 + B_z^2\right) \tag{16}$$

$$= \frac{1}{2}\left(|\vec{E}|^2 - |\vec{B}|^2\right). \tag{17}$$

Question 3 (5 points).

Moore Problem 5.5
Solutions:

(a) Lines of constant u and w looks like this:

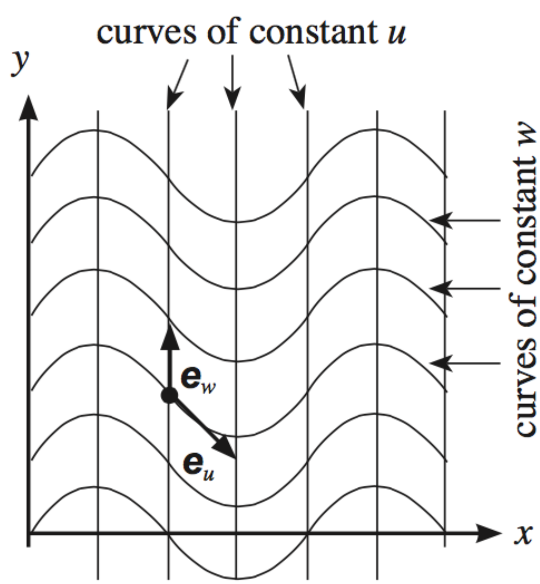

Figure 1: Lines of constant u and w in cartesian coordinates.

(b) The metric in the primed (u, w) coordinates is

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \eta_{\mu\nu}, \tag{18}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The inverse relation between the cartesian coordinates and the (u, w) coordinates are

$$x = u, \qquad y = w + A\sin{(bu)}. \tag{19}$$

Let's work out each component separately.

$$g'_{uu} = \left(\frac{\partial x}{\partial u}\right)^2 \eta_{xx} + \left(\frac{\partial y}{\partial u}\right)^2 \eta_{yy} \tag{20}$$

$$= 1 + A^2 b^2 \cos^2{(bu)} \tag{21}$$

$$g'_{uw} = g'_{wu} = \frac{\partial x}{\partial u}\frac{\partial x}{\partial w}\eta_{xx} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial w}\eta_{yy} \tag{22}$$

$$= Ab\cos{(bu)} \tag{23}$$

$$g'_{ww} = \left(\frac{\partial x}{\partial w}\right)^2 \eta_{xx} + \left(\frac{\partial y}{\partial w}\right)^2 \eta_{yy} \tag{24}$$

$$= 1 \tag{25}$$

Thus, the metric in the (u, w) coordinate system is

$$g'_{\alpha\beta} = \begin{pmatrix} 1 + A^2 b^2 \cos^2{(bu)} & Ab\cos{(bu)} \\ Ab\cos{(bu)} & 1 \end{pmatrix}. \tag{26}$$

This metric is *not* diagonal.

(c) The transformation property for a vector is

$$v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\mu} v^\mu, \tag{27}$$

with $v^x = v$ and $v^y = 0$. We thus have

$$v'^u = \frac{\partial u}{\partial x} v^x = v \tag{28}$$

$$v'^w = \frac{\partial w}{\partial x} v^x = -Ab\cos{(bx)}v = -Abv\cos{(bvt)}. \tag{29}$$

(d) The inner product $\boldsymbol{v}\cdot\boldsymbol{v}$ should be the same the same in every frame. In the cartesian frame it is of course $\boldsymbol{v}\cdot\boldsymbol{v} = v^2$. In the (u, w) frame, it is given by

$$\boldsymbol{v}\cdot\boldsymbol{v} = g'_{\alpha\beta} v'^\alpha v'^\beta \tag{30}$$

$$= g'_{uu}(v'^u)^2 + 2g'_{uw}v'^u v'^w + g'_{ww}(v'^w)^2 \tag{31}$$

$$= (1 + A^2 b^2 \cos^2{(bu)})v^2 + 2Ab\cos{(bu)}v(-Abv\cos{(bvt)}) + (-Abv\cos{(bvt)})^2 \tag{32}$$

$$= v^2 + A^2 b^2 v^2 \cos^2{(bvt)} - 2A^2 b^2 v^2 \cos^2{(bvt)} + A^2 b^2 v^2 \cos^2{(bvt)} \tag{33}$$

$$= v^2, \tag{34}$$

where we used $u = x = vt$. So, of course, the inner product $\boldsymbol{v} \cdot \boldsymbol{v}$ is the same in the (u, w) frame. v'^w is not a constant since the $\mathbf{e}_{(u)}$ basis vector keeps changing direction as one moves in the (x, y) plane, which should be apparent from Fig. 1. Since the vector $\boldsymbol{v}$ is constant, the component v'^w has to keep changing to compensate for the fact that the $\mathbf{e}_{(u)}$ basis vector changes from point to point.

(e) Since $\boldsymbol{v}$ is a constant vector, then we must have $\boldsymbol{a} = d\boldsymbol{v}/dt = 0$. This is obviously true in the cartesian system. But we know that

$$\frac{dv'^w}{dt} = Ab^2v^2 \sin(bvt) \neq 0. \tag{35}$$

So, if we were to write

$$\boldsymbol{a} \stackrel{?}{=} \frac{dv'^u}{dt}\mathbf{e}_{(u)} + \frac{dv'^w}{dt}\mathbf{e}_{(w)} = \frac{dv'^w}{dt}\mathbf{e}_{(w)} \neq 0, \tag{36}$$

we would get something nonzero, in contradiction with the fact that $\boldsymbol{v}$ is a constant vector. To resolve this, we need to remember that $\mathbf{e}_{(u)}$ is not a constant vector. This means that the acceleration is really given by

$$\boldsymbol{a} = \frac{d}{dt}\left(v'^u\mathbf{e}_{(u)} + v'^w\mathbf{e}_{(w)}\right) \tag{37}$$

$$= \frac{dv'^u}{dt}\mathbf{e}_{(u)} + v'^u\frac{d\mathbf{e}_{(u)}}{dt} + \frac{dv'^w}{dt}\mathbf{e}_{(w)} + v'^w\frac{d\mathbf{e}_{(w)}}{dt} \tag{38}$$

$$= v'^u\frac{d\mathbf{e}_{(u)}}{dt} + \frac{dv'^w}{dt}\mathbf{e}_{(w)}. \tag{39}$$

Now, both of the terms in the last line are not zero. In fact, they are equal and opposite, resulting in $\boldsymbol{a} = 0$ as it should. So indeed, dv'^w/dt is not the w-component of the acceleration.