# Physics 480/581
# General Relativity

## Homework Assignment 4 Solutions

**Question 1** (2 points).

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Given any two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, one defines the second-rank tensor $\boldsymbol{T} = \boldsymbol{p} \otimes \boldsymbol{q}$ (where $\otimes$ denotes the tensor product) to be a "machine" that takes in two one-forms (dual vectors) $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ and returns a number

$$\boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{\lambda}) = (\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\lambda} \cdot \boldsymbol{q}), \tag{1}$$

where $\boldsymbol{\sigma} \cdot \boldsymbol{p} = \sigma_\mu p^\mu$ and similarly for $\boldsymbol{\lambda} \cdot \boldsymbol{q}$.

Show that the components of $\boldsymbol{T} = \boldsymbol{p} \otimes \boldsymbol{q}$ are the product of the components of $\boldsymbol{p}$ and $\boldsymbol{q}$

$$T^{\mu\nu} = p^\mu q^\nu, \qquad T_\mu{}^\nu = p_\mu q^\nu, \qquad T_{\mu\nu} = p_\mu q_\nu. \tag{2}$$

**Solution:**
The first thing to realize is that

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} = \sigma_\mu p^\mu = g_{\mu\alpha} \sigma^\alpha p^\mu = \sigma^\alpha p^\mu g_{\mu\alpha} = \sigma^\alpha p_\alpha = \sigma^\mu p_\mu, \tag{3}$$

and similarly for $\boldsymbol{\lambda} \cdot \boldsymbol{q}$. Thus,

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{\sigma}, \boldsymbol{\lambda}) &= (\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\lambda} \cdot \boldsymbol{q}) \\ &= \sigma_\mu p^\mu \lambda_\nu q^\nu = p^\mu q^\nu \sigma_\mu \lambda_\nu = T^{\mu\nu} \sigma_\mu \lambda_\nu && (4) \\ &= \sigma^\mu p_\mu \lambda_\nu q^\nu = p_\mu q^\nu \sigma^\mu \lambda_\nu = T_\mu{}^\nu \sigma^\mu \lambda_\nu && (5) \\ &= \sigma^\mu p_\mu \lambda^\nu q_\nu = p_\mu q_\nu \sigma^\mu \lambda^\nu = T_{\mu\nu} \sigma^\mu \lambda^\nu, && (6) \end{aligned}$$

from which we get

$$T^{\mu\nu} = p^\mu q^\nu, \qquad T_\mu{}^\nu = p_\mu q^\nu, \qquad T_{\mu\nu} = p_\mu q_\nu. \tag{7}$$

**Question 2** (3 points).

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Let $A_{\mu\nu}$ be an antisymmetric tensor such that $A_{\mu\nu} = -A_{\nu\mu}$, and let $S^{\mu\nu}$ be a symmetric tensor such that $S^{\mu\nu} = S^{\nu\mu}$.

**(a)** Show that the trace of an antisymmetric tensor, $A^\mu{}_\mu$, is always zero.
**Solution:**
We have

$$A^\mu{}_\mu = g^{\mu\nu} A_{\nu\mu} = g^{\nu\mu} A_{\nu\mu}. \tag{8}$$

Now the inverse metric is a symmetric tensor such that $g^{\mu\nu} = g^{\nu\mu}$. Per the result of part (a), we know that the contraction of a symmetric tensor with an antisymmetric tensor is always zero. Thus, $A^\mu{}_\mu = 0$.

**(b)** For an arbitrary tensor $V_{\mu\nu}$, show what

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} = V^{[\mu\nu]}A_{\mu\nu}, \qquad V^{\mu\nu}S_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)S_{\mu\nu} = V^{(\mu\nu)}S_{\mu\nu} \tag{9}$$

**Solution:**
For an antisymmetric tensor, first note that

$$A_{\mu\nu} = \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right). \tag{10}$$

We thus have

$$\begin{aligned} V^{\mu\nu}A_{\mu\nu} &= V^{\mu\nu}\frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} - V^{\mu\nu}A_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} - V^{\nu\mu}A_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu}, \end{aligned} \tag{11}$$

where in the third line we have relabelled $\mu \to \nu$ and $\nu \to \mu$ in the last term. For a symmetric tensor, we can always write

$$S_{\mu\nu} = \frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right). \tag{12}$$

We thus have

$$\begin{aligned} V^{\mu\nu}S_{\mu\nu} &= V^{\mu\nu}\frac{1}{2}\left(S_{\mu\nu} + S_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}S_{\mu\nu} + V^{\mu\nu}S_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\mu\nu}\right) \\ &= \frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)S_{\mu\nu}, \end{aligned} \tag{13}$$

where in the third line we have relabelled $\mu \to \nu$ and $\nu \to \mu$ in the last term.

**(c)** Show that a second-rank tensor can always be reconstructed from it's symmetric and anti-symmetric parts,

$$V_{\mu\nu} = V_{(\mu\nu)} + V_{[\mu\nu]}. \tag{14}$$

**Solution:**
Direct calculation:

$$\begin{aligned} V_{(\mu\nu)} + V_{[\mu\nu]} &= \frac{1}{2}\left(V_{\mu\nu} + V_{\nu\mu}\right) + \frac{1}{2}\left(V_{\mu\nu} - V_{\nu\mu}\right) \\ &= \frac{1}{2}\left(V_{\mu\nu} + V_{\mu\nu}\right) + \frac{1}{2}\left(V_{\nu\mu} - V_{\nu\mu}\right) \\ &= V_{\mu\nu}. \end{aligned} \tag{15}$$

**Question 3** (4 points).

**[This question has a lot of text, but the calculations involved are rather short. This is really about learning important aspects of coordinate basis vectors.]** We have discussed that vectors defined at some point $p$ on some manifold $M$ lives in the tangent space $T_pM$. We also introduced basis vectors for this tangent space $\{\mathbf{e}_{(\mu)}\}$, such that vectors can be written as $\boldsymbol{p} = p^\mu \mathbf{e}_{(\mu)}$. Here, we would like to formalize the notion of this tangent space.

Let's consider a point (event) $p$ on some manifold $M$, and imagine that we have a smooth function $f : M \to \mathbb{R}$ defined on this manifold. Then, consider the set of all parametrized curves (worldlines) $\{x_i^\mu(\lambda_i)\}$ passing through $p$, where $i$ is labeling the different curves. We notice that all these curves passing through $p$ each define an operator through this space called the *directional derivative*, which maps $f \to df/d\lambda_i$ at $p$. **The tangent space $T_pM$ can be identified with the space of directional derivative operators alongs curves at $p$.**

To show that this is true, we must demonstrate that the space of directional derivative is indeed a vector space, and that this is the vector space we want. Showing that this is a vector space is straightforward. Take two directional derivative operators $d/d\lambda_1$ and $d/d\lambda_2$ representing derivatives through two curves $x_1^\mu(\lambda_1)$ and $x_2^\mu(\lambda_2)$ at $p$. We can certainly build a new directional derivative operator at $p$

$$\frac{d}{d\eta} = a\frac{d}{d\lambda_1} + b\frac{d}{d\lambda_2}, \tag{16}$$

where $a, b$ are real numbers. So, directional derivatives behave like vectors. The question is whether $d/d\eta$ itself is an actual directional derivative operator. To answer this question, we need to show that $d/d\eta$ behaves like a standard derivative operators, that is, that it acts linearly on function and obeys the product rule. The linearity is manifest, but we need to show that it satisfied the product rule.

**(a)** Show that for two functions smooth $f$ and $g$ defined on $M$, we indeed have:

$$\frac{d}{d\eta}(fg) = g\frac{df}{d\eta} + f\frac{dg}{d\eta}. \tag{17}$$

**Solutions:**
A direct calculation yields

$$\begin{aligned}\frac{d}{d\eta}(fg) &= \left(a\frac{d}{d\lambda_1} + b\frac{d}{d\lambda_2}\right)(fg)\\ &= g\left(a\frac{df}{d\lambda_1} + b\frac{df}{d\lambda_2}\right) + f\left(a\frac{dg}{d\lambda_1} + b\frac{dg}{d\lambda_2}\right)\\ &= g\frac{df}{d\eta} + f\frac{dg}{d\eta}.\end{aligned} \tag{18}$$

**(b)** Having established that the space of directional derivatives at point $p$ is indeed a vector space, we need to find a basis for this space. A simple application of the chain rule for an arbitrary smooth function $f$ defined on $M$ yields

$$\frac{df}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} = \frac{dx^\mu}{d\lambda}\partial_\mu f, \tag{19}$$

where we have used the common notation $\partial_\mu \equiv \partial/\partial x^\mu$. Since this is true for an arbitrary function $f$, we have

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu. \tag{20}$$

The above says that we can expand any directional derivative operator in a basis spanned by the partial derivatives $\{\partial_\mu\}$. If $T_pM$ is identified as the space of directional derivatives at $p$, it follows that a natural basis for this tangent space are the partial derivatives, i.e. $\mathbf{e}_{(\mu)} = \partial_\mu$. With this choice, we naturally have the right number of basis vectors (one per coordinate), and the correct transformation law for basis vectors under the coordinate transform $x^\mu \to x'^\mu$

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu, \tag{21}$$

which follows from the chain rule. This is identical to what we saw in class

$$\mathbf{e}'_{(\mu)} = \frac{\partial x^\nu}{\partial x'^\mu}\mathbf{e}_{(\nu)}. \tag{22}$$

Thus, the partial derivatives $\{\partial_\mu\}$ form a natural basis for $T_pM$. Then, any vector $\boldsymbol{u}$ and $\boldsymbol{v}$ at point $p$ can be expanded in this basis, $\boldsymbol{u} = u^\mu\partial_\mu$ and $\boldsymbol{v} = v^\mu\partial_\mu$. Now, you may be frowning at this. Are vectors differential operators?? Yes! Every vector we write down defines a directional derivative operator at point $p$. The action of a vector on a function $f$ is rather natural

$$\boldsymbol{u}(f) = u^\mu\partial_\mu f, \tag{23}$$

which is indeed the directional derivative of $f$ along the vector $\boldsymbol{u}$. So basically, you need to rewire your brain to think that vectors are machines that are eagerly awaiting to take directional derivatives of any functions they encounter. But we don't have to restrict ourselves to functions, we can also take directional derivatives of other vectors. For instance, consider the commutator $[\boldsymbol{u}, \boldsymbol{v}]$ (also known as the Lie bracket), which is defined via

$$[\boldsymbol{u}, \boldsymbol{v}](f) \equiv \boldsymbol{u}(\boldsymbol{v}(f)) - \boldsymbol{v}(\boldsymbol{u}(f)). \tag{24}$$

Show that the components of this commutator are

$$[\boldsymbol{u}, \boldsymbol{v}]^\mu = u^\lambda\partial_\lambda v^\mu - v^\lambda\partial_\lambda u^\mu. \tag{25}$$

**Solutions:**
First note that $[\boldsymbol{u}, \boldsymbol{v}]$ is a vector, and can thus be written as

$$[\boldsymbol{u}, \boldsymbol{v}] = [\boldsymbol{u}, \boldsymbol{v}]^\mu\partial_\mu. \tag{26}$$

To derive this, let's apply the commutator on a scalar function $f$

$$\begin{aligned}
[\boldsymbol{u}, \boldsymbol{v}](f) &= [u^\mu\partial_\mu, v^\nu\partial_\nu]f \\
&= u^\mu\partial_\mu(v^\nu\partial_\nu f) - v^\nu\partial_\nu(u^\mu\partial_\mu f) \\
&= (\partial_\nu f)u^\mu(\partial_\mu v^\nu) + v^\nu u^\mu(\partial_\mu\partial_\nu f) - (\partial_\mu f)v^\nu(\partial_\nu u^\mu) - u^\mu v^\nu(\partial_\nu\partial_\mu f) \\
&= (\partial_\nu f)u^\mu(\partial_\mu v^\nu) - (\partial_\mu f)v^\nu(\partial_\nu u^\mu) \\
&= (u^\mu(\partial_\mu v^\nu)\partial_\nu - v^\nu(\partial_\nu u^\mu)\partial_\mu)\, f \\
&= \left(u^\lambda(\partial_\lambda v^\mu)\partial_\mu - v^\lambda(\partial_\lambda u^\mu)\partial_\mu\right) f \\
&= \left(u^\lambda(\partial_\lambda v^\mu) - v^\lambda(\partial_\lambda u^\mu)\right)\partial_\mu f = [\boldsymbol{u}, \boldsymbol{v}]^\mu\partial_\mu f,
\end{aligned} \tag{27}$$

from which we get

$$[\boldsymbol{u}, \boldsymbol{v}]^\mu = u^\lambda \partial_\lambda v^\mu - v^\lambda \partial_\lambda u^\mu. \tag{28}$$

**(c)** Having found a natural basis for $T_pM$, the question is now to find a basis for the dual space $T_p^*M$. Consider the total differential (usually referred to as the gradient) of a function $f$

$$\mathrm{d}f = (\partial_\mu f)\mathrm{d}x^\mu \tag{29}$$

This is reminiscent of a dual vector (or one-form) like $\boldsymbol{\omega} = \omega_\mu \mathbf{e}^{(\mu)}$, with component $\partial_\mu f$ and basis vectors $\mathrm{d}x^\mu$. Demonstrate that the gradient above is indeed a dual vector by showing that both components and basis vectors transform in the proper way (using the transformations we saw in class). This means that any dual vector (one-form) can be written as $\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu$. Because of this, one-forms (and higher order forms) play an important role in integration over manifolds.

**Solutions:**
We need to show that $\partial_\mu f$ transforms like a dual vector, and that $\mathrm{d}x^\mu$ transforms like a dual basis vector. Under the coordinate change $x^\mu \to x'^\mu$ and using the chain rule, we obtain

$$\partial'_\mu f = \frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu f, \tag{30}$$

which in indeed the transformation law of a dual vector. For the dual basis vectors, we can again use the chain rule,

$$\mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu, \tag{31}$$

which is indeed the transformation law for dual basis vectors.