PHYS 480/581
General Relativity

Homework Assignment 5

Due date: Wednesday 02/21/2024 5pm, submitted electronically on UNM Canvas

Question 1 (4 points).

Consider adding to the Lagrangian of electromagnetism an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \tilde{F}_{\rho\sigma} F^{\rho\sigma}$. Here, $\tilde{F}_{\rho\sigma}$ is called the Hodge dual of the standard electromagnetic field strength $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$, and $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol.

(a) Express $\mathcal{L}'$ in terms of the $\vec{E}$ and $\vec{B}$ fields.

(b) Using the Euler-Lagrange equation for the vector potential A^μ

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\nu)} \right) = 0, \tag{1}$$

and the solution from the extra problem #4, show that including $\mathcal{L}'$ does not affect Maxwell's equations. Here, $\mathcal{L}_{\text{em}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}'$. Can you provide a deep reason why?

Question 2 (4 points).

A quantity that we will need in the near future is the determinant of the metric $\det(g)$. In a 4-dimensional spacetime, this determinant can be computed via the relation

$$\tilde{\epsilon}^{\mu\nu\alpha\beta} g_{\mu\gamma} g_{\nu\delta} g_{\alpha\sigma} g_{\beta\rho} = \det(g)\, \tilde{\epsilon}_{\gamma\delta\sigma\rho}, \tag{2}$$

which is just the standard expression for the determinant of a 4×4 matrix.

(a) Show that for a diagonal metric, the above expression simply gives that $\det(g)$ is just the product of the diagonal elements of $g_{\mu\nu}$.

(b) While $\det(g)$ is a scalar function, it is not a *Lorentz* scalar, meaning that it takes different values in different inertial reference frames. Using the transformation for the Levi-Civita symbol under a coordinate transform $x^\mu \to x'^\mu$

$$\tilde{\epsilon}'_{\alpha\beta\gamma\delta} = \left| \frac{\partial x'}{\partial x} \right| \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} \frac{\partial x^\rho}{\partial x'^\delta} \tilde{\epsilon}_{\mu\nu\sigma\rho}, \tag{3}$$

show that

$$\det(g') = \left| \frac{\partial x'}{\partial x} \right|^{-2} \det(g), \tag{4}$$

where $\left| \frac{\partial x'}{\partial x} \right|$ is the determinant of the Jacobian matrix for the coordinate transformation $x^\mu \to x'^\mu$.

(c) Use Eqs. (3) and (4) above to show that

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{\det(g)}\tilde{\epsilon}_{\alpha\beta\gamma\delta} \tag{5}$$

is an actual tensor, unlike $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$.