Physics 480/581
General Relativity

Homework Assignment 5 Solutions

Question 1 (4 points).

Consider adding to the Lagrangian of electromagnetism an additional term of the form $\mathcal{L}' = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = \tilde{F}_{\rho\sigma} F^{\rho\sigma}$. Here, $\tilde{F}_{\rho\sigma}$ is called the Hodge dual of the standard electromagnetic field strength $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$, and $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol.

(a) Express $\mathcal{L}'$ in terms of the $\vec{E}$ and $\vec{B}$ fields.
Solutions:
Let's use Moore's convention for the electromagnetic field strength

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{1}$$

Let's compute each of the components of $\tilde{F}_{\rho\sigma} = \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\mu\nu}$. Since $\tilde{F}_{\rho\sigma}$ is antisymmetric, we know it's diagonal elements are zero. Its time-space components are

$$\begin{aligned} \tilde{F}_{0i} &= \tilde{\epsilon}_{\mu\nu 0i} F^{\mu\nu} \\ &= \tilde{\epsilon}_{jk0i} F^{jk} + \tilde{\epsilon}_{kj0i} F^{kj}, \qquad i \neq j \neq k, \qquad \text{(no sum on } j, k) \\ &= 2\tilde{\epsilon}_{jk0i} F^{jk}, \qquad i \neq j \neq k, \qquad \text{(no sum on } j, k) \end{aligned} \tag{2}$$

where we used Latin indices (i, j, k) to represent spatial coordinates. We can write down the 3 possibilities here

$$\tilde{F}_{01} = 2\tilde{\epsilon}_{2301} F^{23} = 2B_x, \quad \tilde{F}_{02} = 2\tilde{\epsilon}_{1302} F^{13} = 2(-1)(-B_y) = 2B_y, \quad \tilde{F}_{03} = 2\tilde{\epsilon}_{1203} F^{12} = 2B_z. \tag{3}$$

By antisymmetry, $\tilde{F}_{i0} = -\tilde{F}_{0i}$. For the space-space components, we have

$$\begin{aligned} \tilde{F}_{ij} &= \tilde{\epsilon}_{\mu\nu ij} F^{\mu\nu} \\ &= \tilde{\epsilon}_{0kij} F^{0k} + \tilde{\epsilon}_{k0ij} F^{k0}, \qquad i \neq j \neq k, \qquad \text{(no sum on } k) \\ &= 2\tilde{\epsilon}_{0kij} F^{0k}, \qquad i \neq j \neq k, \qquad \text{(no sum on } k). \end{aligned} \tag{4}$$

Writing down the three possibilities

$$\tilde{F}_{12} = 2\tilde{\epsilon}_{0312} F^{03} = 2E_z, \quad \tilde{F}_{13} = 2\tilde{\epsilon}_{0213} F^{02} = -2E_y \qquad \tilde{F}_{23} = 2\tilde{\epsilon}_{0123} F^{01} = 2E_x. \tag{5}$$

Thus,

$$\tilde{F}_{\mu\nu} = 2 \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{6}$$

We can now work out the contraction $\tilde{F}_{\mu\nu}F^{\mu\nu}$

$$\begin{aligned}\tilde{F}_{\mu\nu}F^{\mu\nu} &= 2\tilde{F}_{0i}F^{0i} + 2\tilde{F}_{ij}F^{ij}, \qquad j > i \text{ in last term} \\ &= 2(2B_xE_x + 2B_yE_y + 2B_zE_z) + 2(2E_zB_z + 2E_yB_y + 2E_xB_x) \\ &= 8\vec{E}\cdot\vec{B}, \end{aligned} \tag{7}$$

where we used the antisymmetry to write, e.g., $\tilde{F}_{0i}F^{0i} + \tilde{F}_{i0}F^{i0} = 2\tilde{F}_{0i}F^{0i}$.

(b) Using the Euler-Lagrange equation for the vector potential A^μ

$$\frac{\partial \mathcal{L}_{\text{em}}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\nu)}\right) = 0, \tag{8}$$

and the solution from problem 3 in Homework 4, show that including $\mathcal{L}'$ does not affect Maxwell's equations. Here, $\mathcal{L}_{\text{em}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}'$. Can you provide a deep reason why?

Solutions:

When working this kind of Euler-Lagrange problems, we have to remember that A_μ and $\partial_\nu A_\mu$ are considered independent variables. We clearly have that

$$\frac{\partial \mathcal{L}'}{\partial A_\nu} = 0 \tag{9}$$

since $\mathcal{L}'$ depends only on the derivatives of A_μ. So, if we can show that

$$\partial_\mu\left(\frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)}\right) = 0, \tag{10}$$

then $\mathcal{L}'$ won't make any contribution to Maxwell's equations. We have

$$\begin{aligned}\frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} &= \frac{\partial \tilde{F}_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} + \tilde{F}_{\rho\sigma}\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \\ &= \tilde{\epsilon}_{\alpha\beta\rho\sigma}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} + \tilde{\epsilon}_{\alpha\beta\rho\sigma}F^{\alpha\beta}\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \\ &= \tilde{\epsilon}_{\alpha\beta\rho\sigma}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} + \tilde{\epsilon}_{\rho\sigma\alpha\beta}F^{\alpha\beta}\frac{\partial F^{\rho\sigma}}{\partial(\partial_\mu A_\nu)}, \end{aligned} \tag{11}$$

since $\tilde{\epsilon}_{\alpha\beta\rho\sigma} = \tilde{\epsilon}_{\rho\sigma\alpha\beta}$ (even number of permutations). We can relabel the dummy indices to make both terms exactly the same

$$\begin{aligned}\frac{\partial \mathcal{L}'}{\partial(\partial_\mu A_\nu)} &= 2\tilde{\epsilon}_{\alpha\beta\rho\sigma}\frac{\partial F^{\alpha\beta}}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} \\ &= 2\tilde{\epsilon}_{\alpha\beta\rho\sigma}\eta^{\alpha\gamma}\eta^{\beta\epsilon}\frac{\partial F_{\gamma\epsilon}}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} \\ &= 2\tilde{\epsilon}_{\alpha\beta\rho\sigma}\eta^{\alpha\gamma}\eta^{\beta\epsilon}\left(\delta^\mu_\gamma\delta^\nu_\epsilon - \delta^\mu_\epsilon\delta^\nu_\gamma\right)F^{\rho\sigma} \\ &= 2\tilde{\epsilon}_{\alpha\beta\rho\sigma}\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}\right)F^{\rho\sigma} \\ &= 4\tilde{\epsilon}_{\alpha\beta\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}F^{\rho\sigma}, \end{aligned} \tag{12}$$

where we used the result from the last homework. We thus have

$$\begin{aligned}
\partial_\mu\left(\frac{\partial\mathcal{L}'}{\partial(\partial_\mu A_\nu)}\right) &= \partial_\mu\left(4\tilde{\epsilon}_{\alpha\beta\rho\sigma}\eta^{\alpha\mu}\eta^{\beta\nu}F^{\rho\sigma}\right) \\
&= 4\eta^{\beta\nu}\tilde{\epsilon}_{\alpha\beta\rho\sigma}\partial^\alpha F^{\rho\sigma} \\
&= -4\eta^{\nu\beta}\tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha F^{\rho\sigma} \\
&= -4\eta^{\nu\beta}\tilde{\epsilon}_{\beta\alpha\rho\sigma}\left(\partial^\alpha\partial^\rho A^\sigma - \partial^\alpha\partial^\sigma A^\rho\right) \\
&= -4\eta^{\nu\beta}\left(\tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha\partial^\rho A^\sigma - \tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha\partial^\sigma A^\rho\right). \qquad (13)
\end{aligned}$$

Since partial derivatives commute ($\partial^\alpha\partial^\rho = \partial^\rho\partial^\alpha$) while $\tilde{\epsilon}_{\beta\alpha\rho\sigma} = -\tilde{\epsilon}_{\beta\rho\alpha\sigma}$, each term in Eq. (13) is of the form a symmetric component multiplying an antisymmetric symbol, which always vanishes:

$$\begin{aligned}
\tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha\partial^\rho A^\sigma &= \tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\rho\partial^\alpha A^\sigma \\
&= \tilde{\epsilon}_{\beta\rho\alpha\sigma}\partial^\alpha\partial^\rho A^\sigma \\
&= -\tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha\partial^\rho A^\sigma, \qquad (14)
\end{aligned}$$

which implies $\tilde{\epsilon}_{\beta\alpha\rho\sigma}\partial^\alpha\partial^\rho A^\sigma = 0$. In the second step, we have relabeled the dummy indices $\alpha \to \rho$ and $\rho \to \alpha$. We thus indeed get

$$\partial_\mu\left(\frac{\partial\mathcal{L}'}{\partial(\partial_\mu A_\nu)}\right) = 0, \qquad (15)$$

and indeed $\mathcal{L}'$ does not contribute to Maxwell's equation. Fundamentally, $\tilde{F}_{\mu\nu}F^{\mu\nu}$ cannot contribute to the equation of motion of the electromagnetic field since it can be written as a *total* derivative. Total derivative terms in the Lagrangian never contribute to the equation of motion since they can be written as surface terms which live at the boundary of spacetime and thus cannot affect the dynamic inside it. Expanding $\tilde{F}_{\mu\nu}F^{\mu\nu}$ in terms of A^μ

$$\begin{aligned}
\tilde{F}_{\rho\sigma}F^{\rho\sigma} &= \tilde{\epsilon}_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} \\
&= \tilde{\epsilon}_{\mu\nu\rho\sigma}\left((\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial^\sigma A^\rho - \partial^\rho A^\sigma)\right) \\
&= \tilde{\epsilon}_{\mu\nu\rho\sigma}\left((\partial^\mu A^\nu)(\partial^\sigma A^\rho) - (\partial^\mu A^\nu)(\partial^\rho A^\sigma) - (\partial^\nu A^\mu)(\partial^\sigma A^\rho) + (\partial^\nu A^\mu)(\partial^\rho A^\sigma)\right). \qquad (16)
\end{aligned}$$

Now note that from the chain rule, we have

$$\partial^\mu(A^\nu\partial^\rho A^\sigma) = (\partial^\mu A^\nu)(\partial^\rho A^\sigma) + A^\nu(\partial^\mu\partial^\rho A^\sigma), \qquad (17)$$

where the first term on the right-hand side is what appears in Eq. (16). So, for instance, the first term in Eq. (16) can then be written as

$$\tilde{\epsilon}_{\mu\nu\rho\sigma}(\partial^\mu A^\nu)(\partial^\sigma A^\rho) = \tilde{\epsilon}_{\mu\nu\rho\sigma}\partial^\mu(A^\nu\partial^\rho A^\sigma) - \tilde{\epsilon}_{\mu\nu\rho\sigma}A^\nu(\partial^\mu\partial^\rho A^\sigma) = \tilde{\epsilon}_{\mu\nu\rho\sigma}\partial^\mu(A^\nu\partial^\rho A^\sigma), \qquad (18)$$

since $\tilde{\epsilon}_{\mu\nu\rho\sigma}A^\nu(\partial^\mu\partial^\rho A^\sigma) = 0$. Thus, Eq. (16) can be written as

$$\begin{aligned}
\tilde{F}_{\rho\sigma}F^{\rho\sigma} &= \tilde{\epsilon}_{\mu\nu\rho\sigma}\left(\partial^\mu(A^\nu\partial^\sigma A^\rho) - \partial^\mu(A^\nu\partial^\rho A^\sigma) - \partial^\nu(A^\mu\partial^\sigma A^\rho) + \partial^\nu(A^\mu\partial^\rho A^\sigma)\right) \\
&= 4\tilde{\epsilon}_{\mu\nu\rho\sigma}\partial^\mu(A^\nu\partial^\sigma A^\rho), \qquad (19)
\end{aligned}$$

where we have used relabeling of the dummy indices to write the last step. Thus, $\tilde{F}_{\rho\sigma}F^{\rho\sigma}$ is indeed a total derivative and cannot contribute to the equation of motion.

Question 2 (4 points).

A quantity that we will need in the near future is the determinant of the metric $\det(g)$. In a 4-dimensional spacetime, this determinant can be computed via the relation

$$\tilde{\epsilon}^{\mu\nu\alpha\beta} g_{\mu\gamma} g_{\nu\delta} g_{\alpha\sigma} g_{\beta\rho} = \det(g)\, \tilde{\epsilon}_{\gamma\delta\sigma\rho}, \tag{20}$$

which is just the standard expression for the determinant of a 4×4 matrix.

(a) Show that for a diagonal metric, the above expression simply gives that $\det(g)$ is just the product of the diagonal elements of $g_{\mu\nu}$.

Solutions:

(Note that for a metric with Lorentzian signature (that for which the time component is negative, say), Eq. (20) should have a minus sign in front of $\det(g)$. This is because $\tilde{\epsilon}^{\mu\nu\alpha\beta} \equiv \mathrm{sign}(\det(g))\tilde{\epsilon}_{\mu\nu\alpha\beta}$. So, for a Lorentzian metric, if $\tilde{\epsilon}_{0123} = 1$, then $\tilde{\epsilon}^{0123} = -1$. In the following, let's assume we have a Euclidean metric with $\mathrm{sign}(\det(g)) = +1$ since Eq. (20) is valid in this case.)

Evaluate the right-hand side of Eq. (20) at specific indices, for instance 0123

$$\begin{aligned} \det(g)\, \tilde{\epsilon}_{0123} = \det(g) &= \tilde{\epsilon}^{\mu\nu\alpha\beta} g_{\mu 0} g_{\nu 1} g_{\alpha 2} g_{\beta 3} \\ &= \tilde{\epsilon}^{0123} g_{00} g_{11} g_{22} g_{33} \\ &= g_{00} g_{11} g_{22} g_{33}. \end{aligned} \tag{21}$$

So, for a diagonal matrix, the determinant is indeed the product of the diagonal elements.

(b) While $\det(g)$ is a scalar function, it is not a *Lorentz* scalar, meaning that it takes different values in different inertial reference frames. Using the transformation for the Levi-Civita symbol under a coordinate transform $x^\mu \to x'^\mu$

$$\tilde{\epsilon}'_{\alpha\beta\gamma\delta} = \left| \frac{\partial x'}{\partial x} \right| \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} \frac{\partial x^\rho}{\partial x'^\delta} \tilde{\epsilon}_{\mu\nu\sigma\rho}, \tag{22}$$

show that

$$\det(g') = \left| \frac{\partial x'}{\partial x} \right|^{-2} \det(g), \tag{23}$$

where $\left| \frac{\partial x'}{\partial x} \right|$ is the determinant of the Jacobian matrix for the coordinate transformation $x^\mu \to x'^\mu$.

Solutions:

Below, we will need the transformation law of the metric under a coordinate transform

$$g'_{\mu\gamma} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\gamma} g_{\alpha\beta} \tag{24}$$

Multiply both sides by $\frac{\partial x'^\mu}{\partial x^\sigma}$,

$$\begin{aligned} \frac{\partial x'^\mu}{\partial x^\sigma} g'_{\mu\gamma} &= \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\gamma} g_{\alpha\beta} \\ &= \delta^\alpha_\sigma \frac{\partial x^\beta}{\partial x'^\gamma} g_{\alpha\beta} \\ &= \frac{\partial x^\beta}{\partial x'^\gamma} g_{\sigma\beta}. \end{aligned} \tag{25}$$

We will use this expression several times below. In the primed frame, we have

$$\begin{aligned}
\det(g')\,\tilde{\epsilon}'_{\gamma\delta\sigma\rho} &= \tilde{\epsilon}'^{\mu\nu\alpha\beta} g'_{\mu\gamma} g'_{\nu\delta} g'_{\alpha\sigma} g'_{\beta\rho} \\
&= \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x'^{\mu}}{\partial x^{\tau_1}} \frac{\partial x'^{\nu}}{\partial x^{\tau_2}} \frac{\partial x'^{\alpha}}{\partial x^{\tau_3}} \frac{\partial x'^{\beta}}{\partial x'^{\tau_3}} \tilde{\epsilon}^{\tau_1\tau_2\tau_3\tau_4} g'_{\mu\gamma} g'_{\nu\delta} g'_{\alpha\sigma} g'_{\beta\rho} \\
&= \left|\frac{\partial x}{\partial x'}\right| \frac{\partial x^{\mu}}{\partial x'^{\gamma}} g_{\tau_1\mu} \frac{\partial x^{\nu}}{\partial x'^{\delta}} g_{\tau_2\nu} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} g_{\tau_3\alpha} \frac{\partial x^{\beta}}{\partial x'^{\rho}} g_{\tau_4\beta} \tilde{\epsilon}^{\tau_1\tau_2\tau_3\tau_4} \\
&= \left|\frac{\partial x}{\partial x'}\right| \det(g) \tilde{\epsilon}_{\mu\nu\alpha\beta} \frac{\partial x^{\mu}}{\partial x'^{\gamma}} \frac{\partial x^{\nu}}{\partial x'^{\delta}} \frac{\partial x^{\alpha}}{\partial x'^{\sigma}} \frac{\partial x^{\beta}}{\partial x'^{\rho}} \\
&= \left|\frac{\partial x}{\partial x'}\right| \det(g) \left|\frac{\partial x}{\partial x'}\right| \tilde{\epsilon}'_{\gamma\delta\sigma\rho} \\
&= \left|\frac{\partial x}{\partial x'}\right|^2 \det(g) \tilde{\epsilon}'_{\gamma\delta\sigma\rho}.
\end{aligned} \tag{26}$$

Thus, for any value of the components of the right and left hand sides

$$\det(g') = \left|\frac{\partial x}{\partial x'}\right|^2 \det(g) = \left|\frac{\partial x'}{\partial x}\right|^{-2} \det(g), \tag{27}$$

since the inverse of the determinant is the determinant of the inverse.

(c) Use Eqs. (22) and (23) above to show that

$$\epsilon_{\alpha\beta\gamma\delta} = \sqrt{\det(g)}\,\tilde{\epsilon}_{\alpha\beta\gamma\delta} \tag{28}$$

is an actual tensor, unlike $\tilde{\epsilon}_{\alpha\beta\gamma\delta}$.

Solutions:

We need to show that $\epsilon_{\alpha\beta\gamma\delta}$ transforms like a tensor. We have

$$\begin{aligned}
\epsilon'_{\alpha\beta\gamma\delta} &= \sqrt{\det(g')}\,\tilde{\epsilon}'_{\alpha\beta\gamma\delta} \\
&= \sqrt{\left|\frac{\partial x}{\partial x'}\right|^2 \det(g)} \left|\frac{\partial x'}{\partial x}\right| \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} \tilde{\epsilon}_{\mu\nu\sigma\rho} \\
&= \sqrt{\det(g)} \left|\frac{\partial x}{\partial x'}\right| \left|\frac{\partial x'}{\partial x}\right| \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} \tilde{\epsilon}_{\mu\nu\sigma\rho} \\
&= \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} \sqrt{\det(g)}\,\tilde{\epsilon}_{\mu\nu\sigma\rho} \\
&= \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} \frac{\partial x^{\sigma}}{\partial x'^{\gamma}} \frac{\partial x^{\rho}}{\partial x'^{\delta}} \epsilon_{\mu\nu\sigma\rho}.
\end{aligned} \tag{29}$$

Thus, $\epsilon_{\alpha\beta\gamma\delta}$ is indeed a tensor.