Physics 480/581
General Relativity

Homework Assignment 6 Solutions

Question 1 (4 points).

Moore Problem 8.6
Solutions:

(a) Using Eq. (8.12) in Moore, the t component of the geodesic equation is

$$\frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) - \frac{1}{2}\partial_t g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{1}$$

Now, since the metric is time-independent, the time derivative appearing in the second term above always vanishes. We are left with

$$\frac{d}{d\tau}\left(-e^{-x/a}\frac{dt}{d\tau}\right) = 0 \Rightarrow e^{-x/a}\frac{dt}{d\tau} = \text{constant} = c. \tag{2}$$

We thus obtain

$$\frac{dt}{d\tau} = ce^{x/a}. \tag{3}$$

(b) The condition $\boldsymbol{u}\cdot\boldsymbol{u} = -1$ can be written as

$$\boldsymbol{u}\cdot\boldsymbol{u} = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{xx}\left(\frac{dx}{d\tau}\right)^2 = -e^{-x/a}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -1. \tag{4}$$

Using Eq. (3) above, we have

$$\left(\frac{dx}{d\tau}\right)^2 = -1 + e^{-x/a}(ce^{x/a})^2 = -1 + c^2e^{x/a}, \tag{5}$$

which thus yield

$$\frac{dx}{d\tau} = \pm\sqrt{c^2e^{x/a} - 1}. \tag{6}$$

(c) Using Eqs. (3) and (6), we have

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dx} = \pm\frac{\sqrt{c^2e^{x/a} - 1}}{ce^{x/a}}, \tag{7}$$

Integrated on both sides, we can write

$$t = \pm\int dx\,\frac{ce^{x/a}}{\sqrt{c^2e^{x/a} - 1}}. \tag{8}$$

This is fairly easy to integrate analytically with the substitution $v = c^2 e^{x/a} - 1$. Then

$$dv = \frac{c^2 e^{x/a}}{a} dx. \tag{9}$$

Thus,

$$t = \pm \frac{a}{c} \int \frac{dv}{\sqrt{v}} = \pm \frac{a}{c} 2\sqrt{v} + C = \pm \frac{2a}{c} \sqrt{c^2 e^{x/a} - 1} + C, \tag{10}$$

where C is a constant of integration. We can fix it using the initial conditions $x = x_0$ at $t = 0$,

$$0 = \pm \frac{2a}{c} \sqrt{c^2 e^{x_0/a} - 1} + C \Rightarrow C = \mp \frac{2a}{c} \sqrt{c^2 e^{x_0/a} - 1}. \tag{11}$$

Now from Eq. (6) above, we know that

$$\frac{dx}{d\tau}(x_0) = \pm \sqrt{c^2 e^{x_0/a} - 1} = u_0. \tag{12}$$

We thus have

$$t(x) = \pm \frac{2a}{c} \left(\sqrt{c^2 e^{x/a} - 1} - u_0 \right). \tag{13}$$

We can now invert this (pick the positive root)

$$\begin{aligned}
\frac{ct}{2a} &= \sqrt{c^2 e^{x/a} - 1} - u_0 \\
\frac{ct}{2a} + u_0 &= \sqrt{c^2 e^{x/a} - 1} \\
\left(\frac{ct}{2a} + u_0 \right)^2 &= c^2 e^{x/a} - 1 \\
\frac{1}{c^2} \left(\frac{ct}{2a} + u_0 \right)^2 + \frac{1}{c^2} &= e^{x/a} \\
\ln \left[\left(\frac{t}{2a} + \frac{u_0}{c} \right)^2 + \frac{1}{c^2} \right] &= \frac{x}{a}
\end{aligned} \tag{14}$$

We thus get

$$x(t) = a \ln \left[\left(\frac{t}{2a} + \frac{u_0}{c} \right)^2 + \frac{1}{c^2} \right]. \tag{15}$$

$$\begin{aligned}
x(0) &= a \ln \left[\left(\frac{u_0}{c} \right)^2 + \frac{1}{c^2} \right] \\
&= a \ln \left[\frac{c^2 e^{x_0/a} - 1 + 1}{c^2} \right] \\
&= a \ln \left[e^{x_0/a} \right] = x_0
\end{aligned} \tag{16}$$

(d) For $x_0 = 0$ at $t = 0$ (which implies $u_0 = 0$), we have

$$x(t) = a \ln \left[\left(\frac{t}{2a} \right)^2 + \frac{1}{c^2} \right]. \tag{17}$$

The constant c is determined by demanding that $x(0) = 0$

$$a \ln \frac{1}{c^2} = -2a \ln c = 0, \Rightarrow c = 1. \tag{18}$$

The resulting trajectory is plotted in Fig. 1.

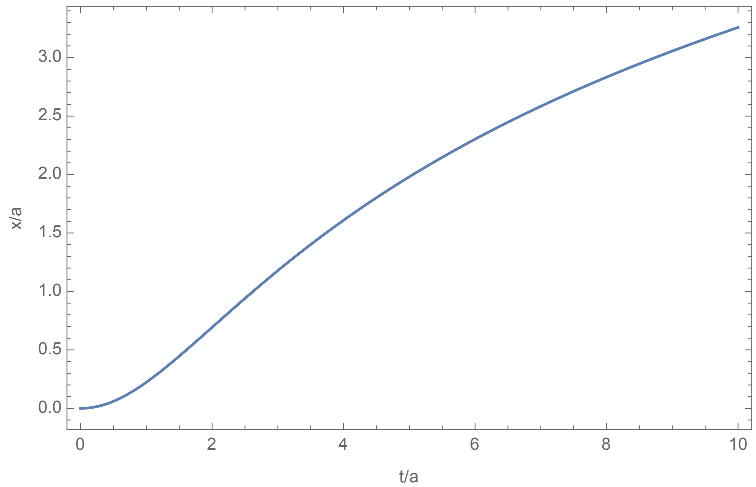

Figure 1: The trajectory of a freely-falling particle.

(e) We got back to Eq. (6) and perform the integral directly

$$\tau(x) = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{c^2 e^{x/a} - 1}} = 2a \left[\tan^{-1} \sqrt{c^2 e^{x/a} - 1} - \tan^{-1} u_0 \right], \tag{19}$$

where we have used Eq. (12) above. For a particle starting at rest at the origin, we have $c = 1$ and $u_0 = 0$, and this simplifies to

$$\tau = 2a \tan^{-1} \sqrt{e^{x/a} - 1}. \tag{20}$$

Let's invert this to get $x(\tau)$

$$\begin{aligned} \tan\left(\frac{\tau}{2a}\right) &= \sqrt{e^{x/a} - 1} \\ \tan^2\left(\frac{\tau}{2a}\right) + 1 &= e^{x/a} \\ \sec^2\left(\frac{\tau}{2a}\right) &= e^{x/a} \end{aligned} \tag{21}$$

and thus

$$x(\tau) = a \ln \left[\sec^2\left(\frac{\tau}{2a}\right) \right] = -2a \ln \left[\cos\left(\frac{\tau}{2a}\right)\right]. \tag{22}$$

Now, from Eq. (3), we have

$$\frac{dt}{d\tau} = e^{x/a} = \sec^2\left(\frac{\tau}{2a}\right), \tag{23}$$

and thus

$$t(\tau) = \int_0^\tau \sec^2\left(\frac{\tau'}{2a}\right) d\tau' = 2a\tan\left(\frac{\tau}{2a}\right). \tag{24}$$

Since both t and x go to infinity as $\tau/2a$ goes to $\pi/2$, the proper time as a maximum value of $\tau = a\pi$.

Question 2 (3 points).

Moore Problem 17.5

Note that the metric in the p, q coordinate system is

$$g'_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & (bq)^{-2} \end{pmatrix}. \tag{25}$$

Solutions:

(a) The Christoffel symbols are given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g'^{\rho\sigma}\left(\partial_\mu g'_{\nu\sigma} + \partial_\nu g'_{\sigma\mu} - \partial_\sigma g'_{\mu\nu}\right), \tag{26}$$

where here $\rho, \sigma, \mu, \nu = p$ or q. Since the only nonvanishing derivative of the metric is $\partial_q g'_{qq} \neq 0$ and the metric is diagonal, the only nonzero Christoffel needs to have q for all its indices

$$\Gamma^q_{qq} = \frac{1}{2} g'^{qq}\left(\partial_q g'_{qq} + \partial_q g'_{qq} - \partial_q g'_{qq}\right) = \frac{1}{2}(bq)^2 \partial_q \frac{1}{(bq)^2} = \frac{1}{2}(bq)^2 \frac{-2}{b^2 q^3} = -\frac{1}{q}. \tag{27}$$

All other Christoffels vanish since $\Gamma^q_{qp} = \Gamma^q_{pq}$, Γ^q_{pp}, Γ^p_{pp}, $\Gamma^p_{pq} = \Gamma^p_{qp}$, and Γ^p_{qq} contain terms of the form $\partial_q g'_{pq}$, $\partial_p g'_{pq}$, $\partial_p g'_{qq}$, $\partial_p g'_{pp}$, and $\partial_q g'_{pp}$, which are all zero.

(b) We have the vector field $\boldsymbol{A}$, which in cartesian coordinates has components

$$\boldsymbol{A} = A^\mu \mathbf{e}_{(\mu)} = Cx\,\mathbf{e}_y. \tag{28}$$

The covariant derivative of $\boldsymbol{A}$ in cartesian coordinates is

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\alpha} A^\alpha = \partial_\nu A^\mu, \tag{29}$$

since all Christoffels are zero in flat cartesian coordinates. We thus have

$$\nabla_x A^x = 0, \qquad \nabla_x A^y = C, \qquad , \nabla_y A^x = 0, \qquad \nabla_y A^y = 0. \tag{30}$$

To find the covariant derivative in the primed frame, we first have to find the components of $\boldsymbol{A}$ in this frame

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu, \tag{31}$$

with $p = x$ and $q = e^{by}$. Then,

$$A'^p = \frac{\partial p}{\partial x} A^x + \frac{\partial p}{\partial y} A^y = \frac{\partial p}{\partial y} A^y = 0\,Cx = 0. \tag{32}$$

$$A'^q = \frac{\partial q}{\partial x}A^x + \frac{\partial q}{\partial y}A^y = \frac{\partial q}{\partial y}A^y = e^{by}bCx = qbCp = Cbpq. \tag{33}$$

We are now ready to compute the covariant derivative of this

$$\nabla_p A'^q = \partial_p A'^q + \Gamma^q_{p\alpha}A'^\alpha = \partial_p A'^q = Cbq, \tag{34}$$

where we have used the fact that $\Gamma^q_{p\alpha} = 0$ for all α.

$$\nabla_q A'^q = \partial_q A'^q + \Gamma^q_{q\alpha}A'^\alpha = \partial_q A'^q + \Gamma^q_{qq}A'^q = Cbp - \frac{1}{q}Cbpq = 0. \tag{35}$$

Of course, $\nabla_\alpha A'^p = 0$ for all α since $A'^p = 0$. It's not surprising that the covariant derivative of a vector has different components in different coordinate systems. It is a bona fide $(1,1)$ tensor that transforms according to the tensor transformation law under a coordinate change, and their components change in performing that transformation. This is what we will do next.

(c) Let's define the $(1,1)$ tensor $M_\nu{}^\mu \equiv \nabla_\nu A^\mu$. Under a coordinate change, this tensor transforms as

$$M'_\nu{}^\mu = \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\beta}M_\alpha{}^\beta. \tag{36}$$

Let's compute each of the four components to this tensor:

$$\begin{aligned} M'_p{}^p &= \frac{\partial x^\alpha}{\partial p}\frac{\partial p}{\partial x^\beta}M_\alpha{}^\beta \\ &= \frac{\partial x}{\partial p}\frac{\partial p}{\partial x}M_x{}^x + \frac{\partial x}{\partial p}\frac{\partial p}{\partial y}M_x{}^y + \frac{\partial y}{\partial p}\frac{\partial p}{\partial x}M_y{}^x + \frac{\partial y}{\partial p}\frac{\partial p}{\partial y}M_y{}^y \\ &= M_x{}^x + (1)(0)M_x{}^y + (0)(1)M_y{}^x + M_y{}^y \\ &= \nabla_x A^x + \nabla_y A^y \\ &= 0. \end{aligned} \tag{37}$$

$$\begin{aligned} M'_q{}^p &= \frac{\partial x^\alpha}{\partial q}\frac{\partial p}{\partial x^\beta}M_\alpha{}^\beta \\ &= \frac{\partial x}{\partial q}\frac{\partial p}{\partial x}M_x{}^x + \frac{\partial x}{\partial q}\frac{\partial p}{\partial y}M_x{}^y + \frac{\partial y}{\partial q}\frac{\partial p}{\partial x}M_y{}^x + \frac{\partial y}{\partial q}\frac{\partial p}{\partial y}M_y{}^y \\ &= (0)(1)M_x{}^x + (0)(0)M_x{}^y + \frac{1}{bq}(1)M_y{}^x + \frac{1}{bq}(0)M_y{}^y \\ &= \frac{1}{bq}\nabla_y A^x \\ &= 0. \end{aligned} \tag{38}$$

$$\begin{aligned} M'_p{}^q &= \frac{\partial x^\alpha}{\partial p}\frac{\partial q}{\partial x^\beta}M_\alpha{}^\beta \\ &= \frac{\partial x}{\partial p}\frac{\partial q}{\partial x}M_x{}^x + \frac{\partial x}{\partial p}\frac{\partial q}{\partial y}M_x{}^y + \frac{\partial y}{\partial p}\frac{\partial q}{\partial x}M_y{}^x + \frac{\partial y}{\partial p}\frac{\partial q}{\partial y}M_y{}^y \\ &= (1)(0)M_x{}^x + (1)bqM_x{}^y + (0)(0)M_y{}^x + (0)bqM_y{}^y \\ &= bq\nabla_x A^y \\ &= Cbq \end{aligned} \tag{39}$$

$$\begin{aligned}
{M'}_q{}^q &= \frac{\partial x^\alpha}{\partial q}\frac{\partial q}{\partial x^\beta}M_\alpha{}^\beta \\
&= \frac{\partial x}{\partial q}\frac{\partial q}{\partial x}M_x{}^x + \frac{\partial x}{\partial q}\frac{\partial q}{\partial y}M_x{}^y + \frac{\partial y}{\partial q}\frac{\partial q}{\partial x}M_y{}^x + \frac{\partial y}{\partial q}\frac{\partial q}{\partial y}M_y{}^y \\
&= (0)(0)M_x{}^x + (0)bqM_x{}^y + \frac{1}{bq}(0)M_y{}^x + M_y{}^y \\
&= \nabla_y A^y \\
&= 0. \qquad (40)
\end{aligned}$$

We indeed retrieve the results from part (b).

Question 3 (2 points).

Show that if we impose the metric compatibility requirement

$$\nabla_\alpha g_{\mu\nu} = 0, \qquad (41)$$

then the connection admits the standard Christoffel form

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \qquad (42)$$

Solutions:
Consider the following three different permutations of the metric compatibility condition

$$\begin{aligned}
\nabla_\rho g_{\mu\nu} &= \partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu}g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu}g_{\mu\sigma} = 0, \\
\nabla_\mu g_{\nu\rho} &= \partial_\mu g_{\nu\rho} - \Gamma^\sigma_{\mu\nu}g_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}g_{\nu\sigma} = 0, \\
\nabla_\nu g_{\rho\mu} &= \partial_\nu g_{\rho\mu} - \Gamma^\sigma_{\nu\rho}g_{\sigma\mu} - \Gamma^\sigma_{\nu\mu}g_{\rho\sigma} = 0. \qquad (43)
\end{aligned}$$

Perform the following operation,

$$\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu} = \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\sigma_{\mu\nu}g_{\sigma\rho} = 0, \qquad (44)$$

where we have used the symmetry of the metric and connection coefficients. We then get

$$\Gamma^\sigma_{\mu\nu}g_{\sigma\rho} = \frac{1}{2}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \qquad (45)$$

Multiplying both sides by $g^{\alpha\rho}$, we obtain

$$\Gamma^\sigma_{\mu\nu}g^{\alpha\rho}g_{\sigma\rho} = \Gamma^\sigma_{\mu\nu}\delta^\alpha_\sigma = \Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right), \qquad (46)$$

which is the Christoffel connection.

Another more indirect way to show this is to take the covariant derivative of the metric, and show that the Christoffel connection leads to the metric compatibility condition:

$$\begin{aligned}
\nabla_\alpha g_{\mu\nu} &= \partial_\alpha g_{\mu\nu} - \Gamma^\sigma_{\alpha\mu} g_{\sigma\nu} - \Gamma^\sigma_{\alpha\nu} g_{\mu\sigma} \\
&= \partial_\alpha g_{\mu\nu} - g_{\sigma\nu}\frac{1}{2} g^{\sigma\beta}\left(\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\alpha\mu}\right) - g_{\mu\sigma}\frac{1}{2} g^{\sigma\beta}\left(\partial_\alpha g_{\nu\beta} + \partial_\nu g_{\beta\alpha} - \partial_\beta g_{\alpha\nu}\right) \\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}\delta^\beta_\nu\left(\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\beta\alpha} - \partial_\beta g_{\alpha\mu}\right) - \frac{1}{2}\delta^\beta_\mu\left(\partial_\alpha g_{\nu\beta} + \partial_\nu g_{\beta\alpha} - \partial_\beta g_{\alpha\nu}\right) \\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}\left(\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\nu\alpha} - \partial_\nu g_{\alpha\mu}\right) - \frac{1}{2}\left(\partial_\alpha g_{\nu\mu} + \partial_\nu g_{\mu\alpha} - \partial_\mu g_{\alpha\nu}\right) \\
&= \partial_\alpha g_{\mu\nu} - \frac{1}{2}(\partial_\alpha g_{\mu\nu} + \partial_\alpha g_{\nu\mu}) - \frac{1}{2}(\partial_\mu g_{\nu\alpha} - \partial_\mu g_{\alpha\nu}) + \frac{1}{2}(\partial_\nu g_{\alpha\mu} - \partial_\nu g_{\mu\alpha}) \\
&= 0,
\end{aligned} \tag{47}$$

where we used the fact that the metric is always symmetric $g_{\mu\nu} = g_{\nu\mu}$.

Question 4 (2 points).

Show that the components of the covariant derivative of a vector A^ν

$$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\alpha} A^\alpha, \tag{48}$$

transform like a tensor under the coordinate transformation $x^\mu \to x'^\mu$.
Solutions:

The key to this is to determine how the Christoffel connection transforms under such a coordinate transformation. Since the connection involves the partial derivative of the metric, let's first determine how this transforms.

$$\begin{aligned}
\partial'_\mu g'_{\nu\sigma} &= \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma} g_{\alpha\beta}\right) \\
&= \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\sigma} g_{\alpha\beta} + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial}{\partial x^\rho} g_{\alpha\beta},
\end{aligned} \tag{49}$$

where we have used the chain rule in the last term. This last term is what we would expect if $\partial_\mu g_{\nu\sigma}$ was a tensor, but since it is not, we also get the two first terms. Combining the three partial derivatives entering the Christoffel connection,

$$\begin{aligned}
\partial'_\mu g'_{\nu\sigma} + \partial'_\nu g'_{\sigma\mu} - \partial'_\sigma g'_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha}\right) + g_{\alpha\beta}\left(\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma}\right. \\
&\quad + \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial^2 x^\beta}{\partial x'^\mu \partial x'^\sigma} + \frac{\partial^2 x^\alpha}{\partial x'^\nu \partial x'^\sigma}\frac{\partial x^\beta}{\partial x'^\mu} + \frac{\partial x^\alpha}{\partial x'^\sigma}\frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\mu} - \frac{\partial^2 x^\alpha}{\partial x'^\sigma \partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \\
&\quad \left. - \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x^\beta}{\partial x'^\sigma \partial x'^\nu}\right) \\
&= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma}\frac{\partial x^\rho}{\partial x'^\mu}\left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha}\right) + 2 g_{\alpha\beta}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}\frac{\partial x^\beta}{\partial x'^\sigma},
\end{aligned} \tag{50}$$

where we have relabeled $\alpha \leftrightarrow \beta$ in several terms and used the fact that $g_{\alpha\beta} = g_{\beta\alpha}$ to cancel several terms (in the large bracket, the last term cancels the third term, and the fifth term the second one;

the two remaining terms are the same after relabeling). The Christoffel connection then transforms as

$$
\begin{aligned}
\Gamma^{\gamma'}_{\mu\nu} &= \frac{1}{2} g'^{\gamma\sigma} \left(\partial'_\mu g'_{\nu\sigma} + \partial'_\nu g'_{\sigma\mu} - \partial'_\sigma g'_{\mu\nu} \right) \\
&= \frac{1}{2} \frac{\partial x'^\gamma}{\partial x^\delta} \frac{\partial x'^\sigma}{\partial x^\epsilon} g^{\delta\epsilon} \left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha} \right) + 2 g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\sigma} \right) \\
&= \frac{1}{2} \frac{\partial x'^\gamma}{\partial x^\delta} \frac{\partial x'^\sigma}{\partial x^\epsilon} \frac{\partial x^\beta}{\partial x'^\sigma} g^{\delta\epsilon} \left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha} \right) + 2 g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \right) \\
&= \frac{1}{2} \frac{\partial x'^\gamma}{\partial x^\delta} \frac{\partial x^\beta}{\partial x^\epsilon} g^{\delta\epsilon} \left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha} \right) + 2 g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \right) \\
&= \frac{1}{2} \frac{\partial x'^\gamma}{\partial x^\delta} \delta^\beta_\epsilon g^{\delta\epsilon} \left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha} \right) + 2 g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \right) \\
&= \frac{1}{2} \frac{\partial x'^\gamma}{\partial x^\delta} g^{\delta\beta} \left(\frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \left(\partial_\rho g_{\alpha\beta} + \partial_\alpha g_{\beta\rho} - \partial_\beta g_{\rho\alpha} \right) + 2 g_{\alpha\beta} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \right) \\
&= \frac{\partial x'^\gamma}{\partial x^\delta} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\alpha}{\partial x'^\nu} \Gamma^\delta_{\rho\alpha} + \frac{\partial x'^\gamma}{\partial x^\delta} \delta^\delta_\alpha \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \\
&= \frac{\partial x'^\gamma}{\partial x^\delta} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\alpha}{\partial x'^\nu} \Gamma^\delta_{\rho\alpha} + \frac{\partial x'^\gamma}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}
\end{aligned} \tag{51}
$$

Note that the last term can also be written as

$$
\begin{aligned}
\frac{\partial x'^\gamma}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} &= \frac{\partial}{\partial x'^\mu} \left(\frac{\partial x'^\gamma}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \right) - \frac{\partial^2 x'^\gamma}{\partial x'^\mu \partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \\
&= . \frac{\partial}{\partial x'^\mu} \left(\delta^\gamma_\nu \right) - \frac{\partial^2 x'^\gamma}{\partial x^\beta \partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu} \\
&= - \frac{\partial^2 x'^\gamma}{\partial x^\beta \partial x^\alpha} \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\mu}.
\end{aligned} \tag{52}
$$

We are now ready to put everything together

$$
\begin{aligned}
\nabla'_\mu A'^\nu &= \partial'_\mu A'^\nu + \Gamma^{\nu'}_{\mu\alpha} A'^\alpha \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial}{\partial x^\rho} \left(\frac{\partial x'^\nu}{\partial x^\sigma} A^\sigma \right) + \left(\frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\alpha} \Gamma^\sigma_{\rho\beta} - \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\alpha} \frac{\partial x^\rho}{\partial x'^\mu} \right) \left(\frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma \right) \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \partial_\rho A^\sigma + \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\alpha} \frac{\partial x'^\alpha}{\partial x^\gamma} \Gamma^\sigma_{\rho\beta} A^\gamma + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} A^\sigma - \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} \frac{\partial x^\sigma}{\partial x'^\alpha} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \partial_\rho A^\sigma + \frac{\partial x'^\nu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \delta^\beta_\gamma \Gamma^\sigma_{\rho\beta} A^\gamma + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} A^\sigma - \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \delta^\sigma_\gamma A^\gamma \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \left(\partial_\rho A^\sigma + \Gamma^\sigma_{\rho\gamma} A^\gamma \right) + \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} A^\sigma - \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial^2 x'^\nu}{\partial x^\rho \partial x^\sigma} A^\sigma \\
&= \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\sigma} \nabla_\rho A^\sigma,
\end{aligned} \tag{53}
$$

and indeed the covariant derivative of a vector transforms like a tensor.