PHYS 480/581
General Relativity

Homework Assignment 8

Due date: Friday 03/22/2024 5pm, submitted electronically on UNM Canvas

Question 1 (5 points).

The Einstein-Hilbert action in n spacetime dimension is given by

$$S_{\mathrm{H}} = \int d^n x \sqrt{-g}\, R = \int d^n x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{1}$$

where R is the Ricci scalar and g is the determinant of the metric. By varying this action with respect to the inverse metric $g^{\mu\nu}$ and setting $\delta S_{\mathrm{H}} = 0$, one can derive Einstein's equation. This variation leads to 3 terms

$$\delta S_{\mathrm{H}} = \int d^n x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} + \int d^n x \sqrt{-g}\, R_{\mu\nu}\, \delta g^{\mu\nu} + \int d^n x R\, \delta \sqrt{-g}. \tag{2}$$

The first term is actually a total derivative (can you show that?) and thus does not contribute to the equation of motion. The second term is already of the form we want (i.e. a variation with respect to the inverse metric). The third term is what we need to focus on.

(a) Using the definition of the inverse metric $g^{\mu\nu}$, show that the variation of the metric and of the inverse metric are related as follows

$$\delta g_{\mu\nu} = -g_{\mu\rho} g_{\nu\sigma} \delta g^{\rho\sigma}. \tag{3}$$

(b) Use the identity $\ln(\det M) = \mathrm{Tr}(\ln M)$ (where M is a square non-singular matrix) and the result from part (a) to show that

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g}\, g_{\mu\nu} \delta g^{\mu\nu}. \tag{4}$$

(c) Use the above results and set $\delta S_{\mathrm{H}} = 0$ to derive Einstein's equation in vacuum

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 0. \tag{5}$$

Question 2 (4 points).

The Lagrangian density for electromagnetism in curved spacetime is

$$\mathcal{L} = \sqrt{-g} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right), \tag{6}$$

where J^μ is the electric four-current and g is the determinant of the metric. Using the definition of the stress-energy tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{7}$$

where $S = \int d^4x\mathcal{L}$ is the action, compute the stress energy tensor for electromagnetism. You may find some of the results from Question 1 useful.

Question 3 (4 points).

Moore Problem 20.10