Physics 480/581
General Relativity

Homework Assignment 8 Solutions

Question 1 (5 points).

The Einstein-Hilbert action in n spacetime dimensions is given by

$$S_{\rm H} = \int d^n x\sqrt{-g}\, R = \int d^n x\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}, \tag{1}$$

where R is the Ricci scalar and g is the determinant of the metric. By varying this action with respect to the inverse metric $g^{\mu\nu}$ and setting $\delta S_{\rm H} = 0$, one can derive Einstein's equation. This variation leads to 3 terms

$$\delta S_{\rm H} = \int d^n x\sqrt{-g}\, g^{\mu\nu}\delta R_{\mu\nu} + \int d^n x\sqrt{-g}\, R_{\mu\nu}\,\delta g^{\mu\nu} + \int d^n x R\,\delta\sqrt{-g}. \tag{2}$$

The first term is actually a total derivative (can you show that?) and thus does not contribute to the equation of motion. The second term is already of the form we want (i.e. a variation with respect to the inverse metric). The third term is what we need to focus on.

(a) Using the definition of the inverse metric $g^{\mu\nu}$, show that the variation of the metric and of the inverse metric are related as follows

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{3}$$

Solutions:
Taking the definition of the inverse metric $g^{\rho\sigma}g_{\sigma\nu} = \delta^\rho_\nu$ and performing a variation on both sides we get

$$\delta g^{\rho\sigma} g_{\sigma\nu} + g^{\rho\sigma}\delta g_{\sigma\nu} = 0. \tag{4}$$

Multiplying this by $g_{\mu\rho}$, we obtain

$$\begin{aligned} g_{\mu\rho}g^{\rho\sigma}\delta g_{\sigma\nu} &= -g_{\mu\rho}g_{\sigma\nu}\delta g^{\rho\sigma} \\ \delta^\sigma_\mu \delta g_{\sigma\nu} &= -g_{\mu\rho}g_{\sigma\nu}\delta g^{\rho\sigma}, \end{aligned} \tag{5}$$

which implies that

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma}. \tag{6}$$

(b) Use the identity $\ln(\det M) = \mathrm{Tr}(\ln M)$ (where M is a square non-singular matrix) and the result from part (a) to show that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu}. \tag{7}$$

Solutions:
Using standard derivative rules, we have

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g, \tag{8}$$

so the problem boils down to finding the variation of the metric determinant δg. This is where the provided identity $\ln(\det M) = \mathrm{Tr}(\ln M)$ becomes useful. Taking the variation on both sides, we get

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{9}$$

Now, identifying M with the matrix form of the $g_{\mu\nu}$ metric, this implies that

$$\frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\mu\nu}. \tag{10}$$

Using Eqs. (10) and (3), we thus get

$$\begin{aligned}
\delta(\sqrt{-g}) &= -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}(-g_{\alpha\mu}g_{\beta\nu}\delta g^{\alpha\beta}) \\
&= -\frac{1}{2}\sqrt{-g}\delta^{\nu}_{\alpha} g_{\beta\nu}\delta g^{\alpha\beta} \\
&= -\frac{1}{2}\sqrt{-g} g_{\beta\alpha}\delta g^{\alpha\beta} \\
&= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\delta g^{\mu\nu},
\end{aligned} \tag{11}$$

where we have relabeled dummy indices in the last step.

(c) Use the above results and set $\delta S_{\mathrm{H}} = 0$ to derive Einstein's equation in vacuum

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0. \tag{12}$$

Solutions:
Setting he variation of the Einstein-Hilbert action to zero, we obtain

$$\begin{aligned}
\delta S_{\mathrm{H}} &= \int d^n x\; \left[\sqrt{-g}R_{\mu\nu}\,\delta g^{\mu\nu} + R\,\delta\sqrt{-g}\right] \\
&= \int d^n x\sqrt{-g}\;\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} \\
&= 0,
\end{aligned} \tag{13}$$

which immediately implies that

$$R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = 0 \tag{14}$$

in vacuum.

Question 2 (4 points).

The Lagrangian density for electromagnetism in curved spacetime is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right), \tag{15}$$

where J^μ is the electric four-current and g is the determinant of the metric. Using the definition of the stress-energy tensor

$$T_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{16}$$

where $S = \int d^4x\mathcal{L}$ is the action, compute the stress energy tensor for electromagnetism. You may find some of the results from Question 1 useful.

Solutions:

Here we are interested in the variation of the action S with respect to the inverse metric $g^{\mu\nu}$. So, we can think of $S[g^{\mu\nu}]$ as a functional (i.e. a function of a function) and we are interested in computing the functional derivative $\delta S/\delta g^{\mu\nu}$. In computing this derivative, we assume that $F_{\mu\nu}$, A_μ and J^μ are constant. First, note that the Lagrangian density depends on the inverse metric in this way

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} + A_\mu J^\mu\right), \tag{17}$$

where the last term is independent of the inverse metric since it is the contraction of a fundamental one-form gauge field A_μ and with the current vector (i.e. not a one-form) J^μ.

The variation of S with respect to $g^{\mu\nu}$ is then

$$\begin{aligned}
\delta S &= \int d^4x\left[\delta(\sqrt{-g})\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) + \sqrt{-g}\left(-\frac{1}{4}(\delta g^{\mu\alpha}g^{\nu\beta} + g^{\mu\alpha}\delta g^{\nu\beta})F_{\alpha\beta}F_{\mu\nu}\right)\right]\\
&= \int d^4x\left[\delta(\sqrt{-g})\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) + \sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{4}(g^{\gamma\beta}F_{\nu\beta}F_{\mu\gamma} + g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu})\right)\right]\\
&= \int d^4x\left[\delta(\sqrt{-g})\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right) + \sqrt{-g}\delta g^{\mu\nu}\left(-\frac{1}{2}(g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu})\right)\right],
\end{aligned} \tag{18}$$

where we have relabeled some dummy indices. Now using Eq. (7), we get

$$\delta S = \int d^4x\sqrt{-g}\delta g^{\mu\nu}\left[-\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + A_\alpha J^\alpha\right) + \left(-\frac{1}{2}g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu}\right)\right]. \tag{19}$$

We thus get

$$\frac{\delta S}{\delta g^{\mu\nu}} = \sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + A_\alpha J^\alpha\right) + \left(-\frac{1}{2}g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu}\right)\right], \tag{20}$$

using the rule of functional differentiation. The stress-energy tensor is then

$$\begin{aligned}
T_{\mu\nu} &= -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\\
&= -2\left[-\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + A_\alpha J^\alpha\right) + \left(-\frac{1}{2}g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu}\right)\right]\\
&= g_{\mu\nu}\left(-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + A_\alpha J^\alpha\right) + g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu}\\
&= -\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma} + g_{\mu\nu}A_\alpha J^\alpha.
\end{aligned} \tag{21}$$

where we have used the antisymmetry of the $F_{\mu\nu}$ and the symmetry of the metric to write $g^{\gamma\alpha}F_{\alpha\mu}F_{\gamma\nu} = g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma}$. Note that the last term involving the electric current is often not included in the electromagnetic stress-energy tensor since it describes the contribution of charged particles (typically included in the matter stress-energy tensor) rather than that of electromagnetic field.

Question 3 (4 points).

Moore Problem 20.10
Solutions:

(a) This problem introduces you to the concept of local orthonormal frame (LOF). We shall denote the basis vector in the LOF as $\hat{\mathbf{e}}_{(a)}$, where we shall use latin indices to denote components of tensors in the LOF. Our standard coordinate basis vectors $\hat{\mathbf{e}}_{(\mu)}$ can be written in terms of the LOF basis vectors as

$$\hat{\mathbf{e}}_{(\mu)} = e_\mu{}^a \hat{\mathbf{e}}_{(a)} \tag{22}$$

where the component of $\hat{\mathbf{e}}_{(\mu)}$ in the direction of LOF basis vector $\hat{\mathbf{e}}_{(a)}$, denoted $e_\mu{}^a$ is referred as the "vielbein" or "tetrad". The key properties of the vielbeins is that they are orthonormal with respect to the standard coordinate basis metric

$$g_{\mu\nu} e^\mu{}_a e^\nu{}_b = \eta_{ab}, \tag{23}$$

where $e^\mu{}_a$ are the inverse vielbeins, defined via $e^\mu{}_a e_\nu{}^a = \delta^\mu_\nu$ and $e_\mu{}^a e^\mu{}_b = \delta^a_b$. Note that Moore denotes the inverse vielbeins as $e^\mu{}_a = (\boldsymbol{o}_a)^\mu$. Using Eq. (23), we can derive an alternative relation between the vielbein and its inverse

$$e_\mu{}^a = g_{\mu\nu}\eta^{ab} e^\nu{}_b \tag{24}$$

Now, if you know the components V^μ of some vector $\mathbf{V}$ in our standard coordinate basis, the components of $\mathbf{V}$ in the LOF basis are simply

$$V^a = e_\mu{}^a V^\mu. \tag{25}$$

Similarly, if you know the stress-energy tensor components in a coordinate basis, its components in the LOF basis will simply be

$$T^{ab} = e_\mu{}^a e_\nu{}^b T^{\mu\nu}, \tag{26}$$

where T^{ab} here is what Moore calls T^{ab}_{obs}. Note that substituting Eq. (24) into Eq. (26) yield Eq. (20.34) in Moore. Now the dominant energy condition (DEC) implies that the vector $\boldsymbol{b}$ is causal if the vector $\boldsymbol{a}$ is also causal, where the components of $\boldsymbol{b}$ are

$$b^\beta = -T^{\beta\nu} g_{\nu\alpha} a^\alpha = -T^{\beta\nu} a_\nu. \tag{27}$$

We want to show that a fluid's four-momentum's density $\pi^a \equiv T^{ta}$ is causal in a LOF, where the latin indices on π and T indicate that these are evaluated in the LOF.

$$\pi^a = T^{ta} = e_\mu{}^t e_\nu{}^a T^{\mu\nu} \tag{28}$$

Now, in the coordinate basis, ${e_\mu}^t$ is timelike since

$$g^{\mu\nu}{e_\mu}^t{e_\nu}^t = \eta^{tt} = -1. \tag{29}$$

The four-velocity of an observer at rest in a LOF is $u^a_{\text{obs}} = (1, 0, 0, 0)$. But the observer's four-velocity in the coordinate basis will also be $u^\mu = (1, 0, 0, 0)$, since she's at rest. Since

$$\begin{aligned} u^t_{\text{obs}} &= {e_\mu}^t u^\mu \\ 1 &= {e_t}^t > 0, \end{aligned} \tag{30}$$

and ${e_\mu}^t$ is thus a causal vector in the coordinate basis. If the DEC is true, then this means that $b^\nu = -T^{\nu\mu}{e_\mu}^t$ is also causal, as per Eq. (27). We this get

$$\pi^a = -{e_\nu}^a b^\nu = -b^a. \tag{31}$$

Since $\boldsymbol{b}$ is causal, we indeed get that $\boldsymbol{\pi}$ is causal.

(b) Let's assume the stress-energy tensor of a perfect fluid which is at rest (by definition) in its LOF, and his given by $T^{ab} = (\rho_0 + p_0)u^a u^b + p_0\eta^{ab}$. Let's compute the time and spatial components of the DEC (Eq. (27))

$$b^t = -T^{tb}\eta_{bc}a^c = -T^{tt}\eta_{tt}a^t = \rho_0 a^t \tag{32}$$

$$b^i = -T^{ib}\eta_{bc}a^c = -T^{ii}\eta_{ii}a^i = -p_0 a^i, \tag{33}$$

where $i = x, y, z$. Now $\boldsymbol{a}$ is a causal vector, which means that

$$a^a a_a = \eta_{ab}a^a a^b = -(a^t)^2 + (a^x)^2 + (a^y)^2 + (a^z)^2 \leq 0, \tag{34}$$

which implies that

$$(a^t)^2 \geq (a^x)^2 + (a^y)^2 + (a^z)^2. \tag{35}$$

Also $a^t > 0$. By the DEC, $\boldsymbol{b}$ is also causal, which means that $b^t > 0$ and

$$b_a b^a = \eta_{ab}b^a b^b = -(b^t)^2 + (b^x)^2 + (b^y)^2 + (b^z)^2 = -(\rho_0 a^t)^2 + p_0^2((a^x)^2 + (a^y)^2 + (a^z)^2) \leq 0. \tag{36}$$

The worse possible case for this equation is when we saturate Eq. (35) (that is, we take the equality). We then get

$$\eta_{ab}b^a b^b = -(\rho_0 a^t)^2 + p_0^2(a^t)^2 = (-\rho_0^2 + p_0^2)(a^t)^2 \leq 0, \tag{37}$$

which implies that

$$\rho_0^2 \geq p_0^2, \tag{38}$$

since $(a^t)^2 > 0$. Now using Eq. (32), we have $\rho_0 > 0$ since both $a^t > 0$ and $b^t > 0$ (by the DEC). We thus get

$$\rho_0 \geq |p_0|. \tag{39}$$

(c) We have

$$b^\mu = -T^{\mu\nu}g_{\nu\alpha}a^\alpha = \Lambda g^{\mu\nu}g_{\nu\alpha}a^\alpha = \Lambda\delta^\mu_\alpha a^\alpha = \Lambda a^\mu, \tag{40}$$

which automatically implies that if $\boldsymbol{a}$ is causal, then $\boldsymbol{b}$ is causal. To see this, note that Λ is positive, which means that $b^t > 0$ if $a^t > 0$ and $b_\mu b^\mu = \Lambda^2 a_\mu a^\mu \leq 0$ if $\boldsymbol{a}$ is causal. Thus, a stress-energy tensor of the form $\Lambda g^{\mu\nu}$ satisfies the DEC.

(d) In part (a), we use the DEC to show that $\pi^a = T^{ta}$ is causal. This means that $\pi^t > 0$. Now, $\pi^t = T^{tt} = \rho_0$, which means that $\rho_0 > 0$. Since this latter statement is the Weak Energy condition (WEC), we just showed that the DEC implies the WEC.