PHYS 480/581
General Relativity

Homework Assignment 9 Solutions

Question 1 (6 points).

Consider the Friedmann-Robertson-Lemaitre-Walker metric given by

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \tag{1}$$

where $a(t)$ is a function of coordinate time to be determined.

(a) Assuming that the stress-energy tensor is dominated by vacuum energy,

$$T_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}, \tag{2}$$

use the Einstein equation to determine $a(t)$.

Solutions:

We first have to work out the left-hand side of the Einstein equation involving the Ricci tensor and scalar. Let's start by computing the Christoffel connections. First, note that the only nonvanishing derivative of metric given in Eq. (1) is

$$\partial_t g_{ij} = 2a\dot{a}\delta_{ij}, \tag{3}$$

where $i, j = 1, 2, 3$ are purely spatial indices here, and $\dot{a} = da/dt$. Since the above metric is diagonal, this means that the only nonzero Christoffel symbols will involve 2 spatial indices and one time index. These are

$$\begin{aligned}\Gamma^t_{ij} &= \frac{1}{2} g^{t\alpha} \left(\partial_i g_{j\alpha} + \partial_j g_{\alpha i} - \partial_\alpha g_{ij}\right) \\ &= \frac{1}{2} g^{tt} \left(-\partial_t g_{ij}\right) \\ &= a\dot{a}\delta_{ij},\end{aligned} \tag{4}$$

$$\begin{aligned}\Gamma^i_{tj} = \Gamma^i_{jt} &= \frac{1}{2} g^{i\alpha} \left(\partial_t g_{j\alpha} + \partial_j g_{\alpha t} - \partial_\alpha g_{tj}\right) \\ &= \frac{1}{2} g^{ik} \left(\partial_t g_{jk}\right) \\ &= \frac{\delta^{ik}}{2a^2} 2a\dot{a}\delta_{jk} \\ &= \frac{\dot{a}}{a}\delta^i_j.\end{aligned} \tag{5}$$

Now the Ricci tensor is given by

$$R_{\sigma\nu} = R^\mu{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\mu_{\nu\sigma} - \partial_\nu \Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{6}$$

Let's consider each component separately,

$$\begin{aligned}
R_{tt} &= \partial_\mu \Gamma^\mu_{tt} - \partial_t \Gamma^\mu_{\mu t} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{tt} - \Gamma^\mu_{t\lambda}\Gamma^\lambda_{\mu t} \\
&= -\partial_t \Gamma^i_{it} - \Gamma^k_{tj}\Gamma^j_{kt} \\
&= -\partial_t \left(\frac{\dot{a}}{a}\delta^i_i\right) - \frac{\dot{a}}{a}\delta^k_j \frac{\dot{a}}{a}\delta^j_k \\
&= -3\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \\
&= -3\frac{\ddot{a}}{a},
\end{aligned} \tag{7}$$

where we have used the fact that $\delta^i_i = 3$ since this is the trace of the 3D Kronecker delta. Also, $\delta^k_j \delta^j_k = \delta^k_k = 3$.

$$\begin{aligned}
R_{ti} &= \partial_\mu \Gamma^\mu_{it} - \partial_i \Gamma^\mu_{\mu t} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{it} - \Gamma^\mu_{i\lambda}\Gamma^\lambda_{\mu t} \\
&= \Gamma^j_{jk}\Gamma^k_{it} - \Gamma^k_{it}\Gamma^t_{kt} - \Gamma^k_{ij}\Gamma^j_{kt} \\
&= 0,
\end{aligned} \tag{8}$$

where we have noticed in the first line that only spatial derivatives of Christoffels appear, which are always zero.

$$\begin{aligned}
R_{ij} &= \partial_\mu \Gamma^\mu_{ji} - \partial_j \Gamma^\mu_{\mu i} + \Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{ji} - \Gamma^\mu_{j\lambda}\Gamma^\lambda_{\mu i} \\
&= \partial_t \Gamma^t_{ji} + \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{jt}\Gamma^t_{ki} - \Gamma^t_{jk}\Gamma^k_{ti} \\
&= \partial_t(a\dot{a}\delta_{ij}) + \frac{\dot{a}}{a}\delta^k_k a\dot{a}\delta_{ij} - \frac{\dot{a}}{a}\delta^k_j a\dot{a}\delta_{ik} - a\dot{a}\delta_{jk}\frac{\dot{a}}{a}\delta^k_i \\
&= \delta_{ij}\left(\dot{a}^2 + a\ddot{a}\right) + 3\dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} \\
&= \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right).
\end{aligned} \tag{9}$$

Finally, the Ricci scalar is

$$\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= g^{tt}R_{tt} + g^{ij}R_{ij} \\
&= -R_{tt} + \frac{1}{a^2}\delta^{ij}R_{ij} \\
&= 3\frac{\ddot{a}}{a} + \frac{1}{a^2}\delta^{ij}\delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) \\
&= 3\frac{\ddot{a}}{a} + \frac{3}{a^2}\left(2\dot{a}^2 + a\ddot{a}\right) \\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].
\end{aligned} \tag{10}$$

Now, let's focus on the tt term of the Einstein equation

$$\begin{aligned}
R_{tt} - \frac{1}{2}Rg_{tt} &= -3\frac{\ddot{a}}{a} + \frac{1}{2}6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \\
&= 3\left(\frac{\dot{a}}{a}\right)^2 \\
&= 8\pi G\, T_{tt}.
\end{aligned} \tag{11}$$

Now since $T_{tt} = \Lambda/(8\pi G)$, the differential equation we are trying to solve is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\Lambda}{3} \Rightarrow \frac{1}{a}\frac{da}{dt} = \pm\sqrt{\frac{\Lambda}{3}}. \tag{12}$$

Choose positive sign for definitiveness. Integrating on both sides leads to

$$\begin{aligned}\int \frac{da}{a} &= \sqrt{\frac{\Lambda}{3}} \int dt \\ \ln a &= \sqrt{\frac{\Lambda}{3}} t + C \\ a(t) &= e^{\sqrt{\Lambda}t/3 + C} \\ &= a_0 e^{\sqrt{\Lambda}t/3},\end{aligned} \tag{13}$$

where C is a constant of integration, and $a_0 = e^C$ is the value of $a(t)$ at $t = 0$. Let's see what the ij component of the Einstein equation tells us

$$\begin{aligned}R_{ij} - \frac{1}{2}Rg_{ij} &= \delta_{ij}\left(2\dot{a}^2 + a\ddot{a}\right) - \frac{1}{2}a^2\delta_{ij}6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \\ &= -\delta_{ij}\left(2a\ddot{a} + \dot{a}^2\right) \\ &= 8\pi G\, T_{ij} \\ &= -\Lambda a^2 \delta_{ij},\end{aligned} \tag{14}$$

which reduces to

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda. \tag{15}$$

Using Eq. (12) above, this can be written as

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3}. \tag{16}$$

Our solution above also satisfies this equation, so we are done and $a(t) = a_0 e^{\sqrt{\Lambda}t/3}$.

(b) Now, assume instead that the stress-energy tensor is dominated by nonrelativistic matter with zero pressure such that

$$T_{\mu\nu} = \begin{pmatrix} \rho_{\mathrm{m}}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{17}$$

where ρ_{m} is the rest-frame energy density of the matter. Using the covariant conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$, show that

$$\rho_{\mathrm{m}} \propto 1/a(t)^3. \tag{18}$$

Solutions:
We have

$$\begin{aligned}\nabla_\mu T^{\mu 0} &= \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\lambda} T^{\lambda 0} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} \\ &= \partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{\mu\lambda} T^{\mu\lambda} \\ &= \partial_0 T^{00} + \Gamma^i_{i0} T^{00} \\ &= \frac{\partial \rho_{\rm m}}{\partial t} + \delta^i_i \frac{\dot{a}}{a} \rho_{\rm m} \\ &= \frac{\partial \rho_{\rm m}}{\partial t} + 3\frac{\dot{a}}{a} \rho_{\rm m} = 0, \end{aligned} \tag{19}$$

where we have used the fact tht $T^{00} = g^{0\mu} g^{0\nu} T_{\mu\nu} = g^{00} g^{00} T_{00} = (-1)(-1)\rho_{\rm m} = \rho_{\rm m}$. We thus have

$$a\frac{\partial \rho_{\rm m}}{\partial t} + 3\dot{a}\rho_{\rm m} = 0, \qquad \Rightarrow \qquad \frac{\partial}{\partial t}\left(a^3 \rho_{\rm m}\right) = 0. \tag{20}$$

This means that $a^3 \rho_{\rm m} =$ constant, which implies that

$$\rho_{\rm m} \propto 1/a(t)^3. \tag{21}$$

(c) Using the solution given in Eq. (18), show that the Einstein equation implies that

$$a(t) \propto t^{2/3}, \tag{22}$$

for a universe dominated by nonrelativistic matter.

Solutions:
Let's write $\rho_{\rm m}(t) = \rho_0/a^3(t)$, where ρ_0 is a constant with units of mass density. Using the tt component of the Einstein equation above, we have

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm m}(t) = \frac{8\pi G}{3}\frac{\rho_0}{a^3} \tag{23}$$

This can be written as

$$\begin{aligned} a\dot{a}^2 &= \frac{8\pi G \rho_0}{3} \\ \sqrt{a}\frac{da}{dt} &= \sqrt{\frac{8\pi G \rho_0}{3}} \\ \int \sqrt{a}da &= \int \sqrt{\frac{8\pi G \rho_0}{3}}\, dt \\ a^{3/2} &= \sqrt{\frac{8\pi G \rho_0}{3}} t, \end{aligned} \tag{24}$$

where we've assumed that a(0) = 0 here. Taking the 2/3 power on both sides we get

$$a(t) = \left(\frac{8\pi G \rho_0}{3}\right)^{1/3} t^{2/3}, \tag{25}$$

and indeed $a(t) \propto t^{2/3}$ for a matter-dominated universe.

Question 2 (5 points).

Moore 23.6 a,c,d,e

Solutions:

- **(a)** We will use the Einstein equation in this form

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right). \tag{26}$$

Since only $T^t{}_t$ and $T^z{}_z$ are nonzero, the trace of the stress-energy tensor is

$$T = T^\mu{}_\mu = T^t{}_t + T^z{}_z = -\sigma(r) - \sigma(r) = -2\sigma(r). \tag{27}$$

Also, since the metric here is diagonal, we have

$$T_{tt} = g_{t\alpha}T^\alpha{}_t = g_{tt}T^t{}_t = -(-\sigma(r)) = \sigma(r), \tag{28}$$

and

$$T_{zz} = g_{z\alpha}T^\alpha{}_z = g_{zz}T^z{}_z = -\sigma(r), \tag{29}$$

and all other $T_{\mu\nu} = 0$. This means that

$$8\pi G\left(T_{tt} - \frac{1}{2}T g_{tt}\right) = 8\pi G(\sigma(r) - \frac{1}{2}(-2\sigma(r))(-1)) = 0, \tag{30}$$

which implies $R_{tt} = 0$. Also,

$$8\pi G\left(T_{zz} - \frac{1}{2}T g_{zz}\right) = 8\pi G\left(-\sigma(r) - \frac{1}{2}(-2\sigma(r))\right) = 0, \tag{31}$$

which implies that $R_{zz} = 0$. So, we are left with

$$R_{rr} = 8\pi G\left(T_{rr} - \frac{1}{2}T g_{rr}\right) = 8\pi G\left(-\frac{1}{2}(-2\sigma(r))\right) = 8\pi G\,\sigma(r), \tag{32}$$

and

$$R_{\phi\phi} = 8\pi G\left(T_{\phi\phi} - \frac{1}{2}T g_{\phi\phi}\right) = 8\pi G(-\frac{1}{2}(-2\sigma(r))f^2) = 8\pi G\, f^2\sigma(r). \tag{33}$$

So, we indeed get that $R_{\phi\phi} = f^2 R_{rr}$.

- **(c)** We need to compute R_{rr} as a function of the unknown function f. The only nonzero derivative of the metric is

$$\partial_r g_{\phi\phi} = \partial_r f^2 = 2f\frac{df}{dr}. \tag{34}$$

Since the metric is diagonal, this means that the only nonvanishing Christoffel connections involve 2 ϕ indices and one r index. These are

$$\begin{aligned}
\Gamma^r_{\phi\phi} &= \frac{1}{2}g^{r\alpha}\left(\partial_\phi g_{\phi\alpha} + \partial_\phi g_{\alpha\phi} - \partial_\alpha g_{\phi\phi}\right) \\
&= -\frac{1}{2}g^{rr}\partial_r g_{\phi\phi} \\
&= -\frac{1}{2}2f\frac{df}{dr} \\
&= -f\frac{df}{dr},
\end{aligned} \tag{35}$$

and

$$\begin{aligned}\Gamma^{\phi}_{r\phi} = \Gamma^{\phi}_{\phi r} &= \frac{1}{2} g^{\phi\alpha} \left(\partial_r g_{\phi\alpha} + \partial_\phi g_{\alpha r} - \partial_\alpha g_{r\phi}\right) \\ &= \frac{1}{2} g^{\phi\phi} \partial_r g_{\phi\phi} \\ &= \frac{1}{2f^2} 2f \frac{df}{dr} \\ &= \frac{1}{f}\frac{df}{dr}. \end{aligned} \tag{36}$$

Now the rr component of the Ricci tensor is

$$\begin{aligned} R_{rr} = R^{\mu}{}_{r\mu r} &= \partial_\mu \Gamma^{\mu}_{rr} - \partial_r \Gamma^{\mu}_{\mu r} + \Gamma^{\mu}_{\mu\lambda}\Gamma^{\lambda}_{rr} - \Gamma^{\mu}_{r\lambda}\Gamma^{\lambda}_{\mu r} \\ &= -\partial_r \Gamma^{\phi}_{\phi r} - \Gamma^{\phi}_{r\phi}\Gamma^{\phi}_{\phi r} \\ &= -\partial_r \left(\frac{1}{f}\frac{df}{dr}\right) - \left(\frac{1}{f}\frac{df}{dr}\right)^2 \\ &= \frac{1}{f^2}\left(\frac{df}{dr}\right)^2 - \frac{1}{f}\frac{d^2 f}{dr^2} - \left(\frac{1}{f}\frac{df}{dr}\right)^2 \\ &= -\frac{1}{f}\frac{d^2 f}{dr^2}. \end{aligned} \tag{37}$$

The rr component of the Einstein equation thus takes the form

$$\frac{d^2 f}{dr^2} = -8\pi G f \sigma. \tag{38}$$

- **(d)** We can rewrite Eq. (38) as

$$\frac{d}{dr}\left(\frac{df}{dr}\right) = -8\pi G f \sigma. \tag{39}$$

Integrating on both sides yields

$$\begin{aligned} \int_0^r dr' \frac{d}{dr'}\left(\frac{df}{dr'}\right) &= -8\pi G \int_0^r dr' f(r')\sigma(r') \qquad (r > r_s) \\ \left.\frac{df}{dr'}\right|_{r'=r} - \left.\frac{df}{dr'}\right|_{r'=0} &= -4G \int_0^{r_s} dr' 2\pi f(r')\sigma(r') \qquad \text{since } \sigma(r' > r_s) = 0 \\ f'(r) - 1 &= -4G\mu \qquad \text{since } f'(0) = 1 \text{ and } \mu \equiv \int_0^{r_s} dr 2\pi\sigma f \\ f'(r) &= 1 - 4G\mu, \qquad \text{for } r > r_s, \end{aligned} \tag{40}$$

where we have adopted the notation $f'(r) = df/dr$.

- **(e)** Integrating Eq. (40) for $r > r_s$, we get

$$f(r) = (1 - 4G\mu)r + K, \qquad r > r_s \tag{41}$$

where K is an integration constant. Now as quoted in part (d), we must have $f(0) = 0$. At the edge of the string, the function f is

$$f(r_s) = (1 - 4G\mu)r_s + K. \tag{42}$$

If K was to be large, we would need to rapidly ramp up $f(r)$ from 0 at $r = 0$ to a large number at $r = r_s$. This would require a large df/dr for $r < r_s$. But, this is impossible since df/dr is bounded as follows in this range

$$\left.\frac{df}{dr}\right|_r = 1 - 4G\int_0^r dr' 2\pi f(r')\sigma(r') \geq 1 - 4G\int_0^{r_s} dr' 2\pi f(r')\sigma(r') = 1 - 4G\mu, \tag{43}$$

where we have assumed that σ anf f are positive over this range. Since $f'(0) = 1$, we have

$$1 \geq f'(r) \geq 1 - 4G\mu, \qquad \text{for } 0 \leq r \leq r_s \tag{44}$$

Thus $f'(r) \leq 1$ in this range, which means that $f(r_s)$ should be at most of order r_s, which means that K should at most also be of order r_s. Thus, for $r \gg r_s$, we can write

$$f(r) \approx (1 - 4G\mu)r. \tag{45}$$

Then show that the circumference of a circle of radius R centered on the cosmic string and at $z = t =$ constant is smaller than $2\pi R$. The spacetime geometry around a cosmic string is thus said to have a *deficit angle*.

Solutions:

The circumference C of such as circle can be compute as follows

$$\begin{aligned} C &= \int_{\text{circle with } t=z=\text{const}} \sqrt{ds^2} \\ &= \int_{\text{circle with } t=z=\text{const}} \sqrt{-dt^2 + dr^2 + f^2 d\phi^2 + dz^2} \\ &= \int \sqrt{f^2 d\phi^2} \\ &= (1 - 4G\mu)R\int_0^{2\pi} d\phi \\ &= (1 - 4G\mu)2\pi R < 2\pi R. \end{aligned} \tag{46}$$