PHYS 480/581: General Relativity
Four-vectors

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I. SPACETIME AS A MANIFOLD

[**Manifolds are locally** $\mathbb{R}^n$] The four-dimensional Minkowski spacetime that we discussed in the context of Special Relativity form a specific mathematical structure called a **manifold**. While we could spend a lot of time discussing the formal definition of a manifold, an intuitive definition is sufficient here: an n-dimensional manifold M is constructed by sewing together sets U that locally looks like $\mathbb{R}^n$. By "looking like $\mathbb{R}^n$", we mean that for any subset U of M, there exists a one-to-one map $\phi : U \to \mathbb{R}^n$ that maps every point within U to a point in n-dimensional Euclidean space (see Fig. 1 below). As you might expect for a physical system, this map needs to have nice properties, like being differentiable. Care must be taken in how we sew the different subset U together to form M, but it's mostly a matter of respecting the standard composition of maps for two overlapping subset of M. In this course, we will be mostly interested in 4D manifolds.

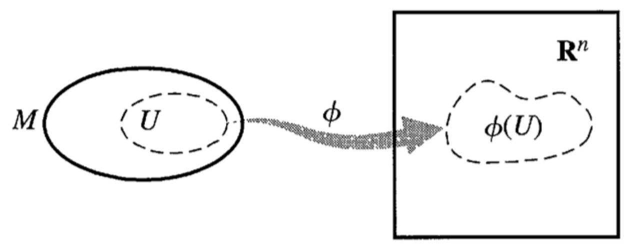

FIG. 1. Map between a subset U of manifold M to a subset of $\mathbb{R}^n$. Reproduced from arXiv:gr-qc/9712019.

II. FOUR-VECTOR: DEFINITION

The above-mentioned 4D spacetime manifold can host different kind of mathematical entities. The first one we will discuss here are **four-vectors**. Physically, four-vectors are defined as any four-component quantity whose components transform according to a Lorentz transformation when we change between different inertial reference frames (we will make this definition more rigorous down the road). For instance, for a four-vector $\boldsymbol{p}$, we can transform it to another inertial frame (moving at constant velocity in the positive x direction, say)

$$\boldsymbol{p}' = \begin{bmatrix} p'^t \\ p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^t \\ p^x \\ p^y \\ p^z \end{bmatrix}. \tag{1}$$

Now, the above definition is a little backward, as Lorentz transformations are *defined* as the family of transformation that leave the magnitude of four-vectors invariant. Going back to the principle of relativity, when we say that the laws of physics are the same in every inertial reference frame, one thing we mean by that is that all inertial observers agree on the magnitude of any four-vector.

[**Four-vectors live in tangent space**] It is important to emphasize that any four-vector is located at a given point in spacetime (rather than stretching from two points in spacetime). The picture you should have in mind is that at every point p on some manifold M, there is a vector space containing all possible four-vectors located at this point; this set of vector is known as the *tangent space* at p, or T_pM (see Fig. 2 below). At a given point in spacetime, four-vectors form a vector space respecting the usual addition and multiplication by scalar rules, that is,

$$(a+b)(\boldsymbol{p}+\boldsymbol{q}) = a\boldsymbol{p} + a\boldsymbol{q} + b\boldsymbol{p} + b\boldsymbol{q}, \tag{2}$$

where a and b are real numbers and $\boldsymbol{p}$ and $\boldsymbol{q}$ are four-vectors.

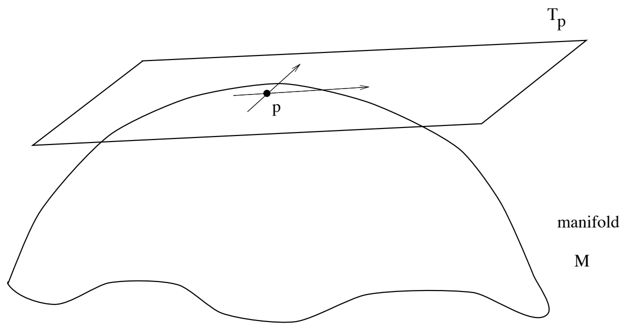

FIG. 2. Tangent space T_p of a two-dimensional manifold at point p. Reproduced from arXiv:gr-qc/9712019.

[**Dual vector space and inner product**] We can also define a dual vector space at point p of spacetime manifold M, T_p^*M. An element $\boldsymbol{\omega}$ of T_p^*M is a map $\boldsymbol{\omega} : T_pM \to \mathbb{R}$ which maps a four-vector to the real numbers. $\boldsymbol{\omega}$ is usually referred to as a "dual vector" or more generally, a one-form. Basically, a one-form is something that takes a four-vector (in this 4D context) and returns a real number. This defines an inner (dot) product for four-vectors.

[**Analogy in 3D Euclidean space**] The above might seem a little esoteric to you if you never seen this before, but you are already familiar with such a construction in 3D Euclidean space: the standard dot product between two 3-vectors. For example, take two real 3-vectors $\vec{u}$ and $\vec{v}$ defined at point p. You know how to compute their dot product, which in cartesian admits the form

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z \in \mathbb{R}. \tag{3}$$

Now in the above, $\vec{u}$ can be thought of as a dual vector living in T_p^*M, while $\vec{v}$ is a vector living in T_pM. We can say that within the above dot product, $\vec{u}$ takes a vector $\vec{v}$ and maps it to a real number. Now, the reason we never really care whether $\vec{u}$ lives in T_p^*M or T_pM when we consider 3D Euclidean vectors is that $T_p^*M = T_pM$ in this case, that is, they are the same vector space. This is because the metric (more on that later) in Euclidean space is just the identity map. So in 3D Euclidean space we can afford to be cavalier about whether an objects live in T_pM or T_p^*M. On the other hand, in SR and GR, we will need to be very careful about whether our objects is a vector or a dual vector (one-form).

[**Inner product of four-vectors**] Using the above construction, the inner (or dot) product $\langle\ ,\ \rangle$ between the dual vector $\boldsymbol{\omega}$ and the vector $\boldsymbol{p}$ can be written as

$$\langle \boldsymbol{\omega}, \boldsymbol{p} \rangle = \boldsymbol{\omega} \cdot \boldsymbol{p} = \boldsymbol{\omega}(\boldsymbol{p}) = -\omega^t p^t + \omega^x p^x + \omega^y p^y + \omega^z p^z, \tag{4}$$

where we have written the product explicitly in terms of the components ω^t, p^t, p^x, etc. of the two vectors in some cartesian basis, assuming that we are in Minkowski space. There is actually quite a bit packed in that last equality as we haven't yet discuss basis vectors for the vector spaces T_pM and T_p^*M, or even how to relate a vector in T_pM to its dual in T_p^*M. For now, take the above as the definition of the inner product between two four-vectors. The crucial point though is that the above inner product is the same in all inertial frame, so all initial observers will agree on its value (see box 3.1).

[**Norm-squared of four-vectors**] The above inner product can be used which can be used to define the norm (square magnitude) of a four-vector

$$\boldsymbol{p} \cdot \boldsymbol{p} = -(p^t)^2 + (p^x)^2 + (p^y)^2 + (p^z)^2. \tag{5}$$

The crucial point is that this vector norm is frame-independent, so all observers will always agree on its value. We will expand more on the structure of this inner product once we introduce the metric.

III. IMPORTANT FOUR-VECTORS

The two most important four-vectors we will encounter are the four-velocity $\boldsymbol{u}$ and the four-momentum $\boldsymbol{p}$.

A. Four-velocity

[**Four-velocity as the tangent vector to a particle's worldline**] We have already seen the proper time $d\tau = \sqrt{-ds^2}$ as the time interval measured by a clock moving along a particle's worldline, where $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ is the spacetime line element (in cartesian coordinates here). Imagine that we use this proper time τ to parameterize the particle's worldline as $\boldsymbol{x}(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$ (this represents a curve in spacetime, where τ is a monotonically increasing parameter along the curve). The four-velocity is defined as the rate of change of this wordline with respect to τ

$$\boldsymbol{u} \equiv \frac{d\boldsymbol{x}}{d\tau} = \begin{bmatrix} dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{bmatrix}, \tag{6}$$

which is vector that is always tangential to the particle's worldline.

[**Normalization of four-velocity**] Note that no matter what the particle worldline is, we *always* have $\boldsymbol{u} \cdot \boldsymbol{u} = -1$, which can easily be proven using $d\tau = \sqrt{-ds^2}$ (see box 3.2). The relationship between three-velocity $\vec{v} = d\vec{x}/dt = (v_x, v_y, v_z)$ and the four-velocity $\boldsymbol{u}$ is

$$\boldsymbol{u} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} \\ \frac{v_x}{\sqrt{1-v^2}} \\ \frac{v_y}{\sqrt{1-v^2}} \\ \frac{v_z}{\sqrt{1-v^2}} \end{bmatrix}, \tag{7}$$

where $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$.

[**Low-speed limit of four-velocity**] Note that for $v \ll 1$, the four-velocity reduces to

$$\boldsymbol{u} \simeq \begin{bmatrix} 1 \\ v_x \\ v_y \\ v_z \end{bmatrix}, \qquad v \ll 1. \tag{8}$$

B. Four-momentum

For an object with a nonzero mass m, the four-momentum is defined as

$$\boldsymbol{p} = m\boldsymbol{u} = \begin{bmatrix} \frac{m}{\sqrt{1-v^2}} \\ \frac{mv_x}{\sqrt{1-v^2}} \\ \frac{mv_y}{\sqrt{1-v^2}} \\ \frac{mv_z}{\sqrt{1-v^2}} \end{bmatrix}. \tag{9}$$

Very generally, the zeroth component of the four-momentum is the energy of the object $p^0 = E$, while the three spatial components form the three-momentum of the object.

[**Relation to three-momentum**] Note that in relativity, the three-momentum of an object is not just $\vec{p} = m\vec{v}$ but rather

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-v^2}}, \tag{10}$$

which of course reduces to $\vec{p} \approx m\vec{v}$ if $v \ll 1$.

[E **and** $\vec{p}$ **are more fundamental than** $\vec{v}$] Note that since v is frame dependent, physicists prefer to think about four-momentum directly in terms of energy and three-momentum, i.e. $\boldsymbol{p} = (E, \vec{p})$. In fact, E and $\vec{p}$ are really the fundamental quantities here (rather than v), and v is actually defined from E and $\vec{p}$

$$\vec{v} \equiv \frac{\vec{p}}{E}. \tag{11}$$

Of course, E and $\vec{p}$ are not frame-independent quantities on their own (i.e. their values is generally different in different inertial frame).

[**Norm-squared of four-momentum**] However, the squared norm of the four-momentum is the same in every frame

$$\boldsymbol{p} \cdot \boldsymbol{p} = (m\boldsymbol{u}) \cdot (m\boldsymbol{u}) = m^2(\boldsymbol{u} \cdot \boldsymbol{u}) = -m^2. \tag{12}$$

Writing $\boldsymbol{p} = (E, \vec{p})$, we then have

$$\boldsymbol{p} \cdot \boldsymbol{p} = -E^2 + p^2 = -m^2 \quad \rightarrow \quad E^2 = m^2 + p^2. \tag{13}$$

In particular, for a massless particle ($m = 0$) such as a photon, we have

$$\boldsymbol{p} \cdot \boldsymbol{p} = 0 \quad \text{and} \quad E = p. \tag{14}$$

[**Four-momentum is conserved**] In any process, like a 2 to 2 scattering event $1 + 2 \rightarrow 3 + 4$ in which two particles collides and two (possibly new) particles come out, four-momentum is always conserved

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_3 + \boldsymbol{p}_4. \tag{15}$$