PHYS 480/581: General Relativity
Arbitrary Coordinates

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I. COORDINATE BASES AND COORDINATE SYSTEMS

A. Introducing coordinate bases

[**Component notation**] Last time, we introduced the component notation for vectors and matrices. For instance, instead of talking about a general vector $\boldsymbol{p}$ or matrix $\boldsymbol{\Lambda}$ (which are objects that exist independently of any choice of basis), we can refer to their components p^μ or $\Lambda^\mu{}_\nu$. In general, explicitly writing down the components (e.g. $p^\mu = (a, b, c, d)$) requires us to choose **a basis** in which these components are expressed. Given a set of basis vectors $\{\mathbf{e}_{(\mu)}\}$ with $\mu = 0, 1, 2, 3$, the vector $\boldsymbol{p}$ can be written as

$$\boldsymbol{p} = p^\mu \mathbf{e}_{(\mu)}. \tag{1}$$

Since the vector $\boldsymbol{p}$ lives in the tangent space T_pM of some point p of some manifold M, $\mathbf{e}_{(\mu)}$ are thus the basis vectors for the vector space T_pM.

[**Analogy with 3-vectors**] Now, we are already familiar with such notation in three-dimensional space. For instance, given some electric field three-vector $\vec{E}$, we can write this vector in a cartesian basis given by $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\} = \{\hat{\mathbf{x}}_i\}_{i=1..3}$ as

$$\vec{E} = E^x \hat{\mathbf{x}} + E^y \hat{\mathbf{y}} + E^z \hat{\mathbf{z}} = E^i \hat{\mathbf{x}}_i, \tag{2}$$

so there is nothing too new here.

[**Coordinate bases**] The above example makes the rather natural choice of choosing a basis for T_pM that is aligned with the coordinate system used to describe the position of spatial points to write the electric field components. This is called a **coordinate basis**. The way this works is that you first choose a coordinate systems to label the region of spacetime of interest. Then, the basis to describe components of vectors (and other tensors) living at points within this region is chosen such that every basis vector $\mathbf{e}_{(\mu)}$ is locally aligned with the direction of the coordinate axes. Since the notion of coordinate basis is intimately linked to the notion of coordinate systems, let's briefly review the latter.

B. Lightning review of coordinate systems

[**Coordinate systems**] We are already pretty familiar with the notation of coordinate systems. In general, it is some kind of organized scheme to attach numbers ("coordinates") to points ("events") in spacetime. You have probably already encounter different kinds of coordinate systems including cartesian, spherical, or cylindrical coordinates. Here, we will of course be interested in these, but also in other more baroque coordinate systems. The only two demands that we make for the coordinate systems that we will use is that:

1. For a small enough region of spacetime, the spacetime should look flat.
2. Our coordinates vary smoothly such that nearby points have nearly the same coordinates.

The above points ensure that we will be focusing our attention on coordinate systems describing points living on a manifold.

C. Coordinate bases

[**Properties of coordinate bases**] While the cartesian case illustrated above is a rather natural example of a coordinate basis, it is important to understand that for a generic coordinate basis $\{\mathbf{e}_{(\mu)}\}$

1. $\mathbf{e}_{(\mu)} \cdot \mathbf{e}_{(\nu)}$ is not necessarily 0 (i.e. the basis vectors don't have to be orthogonal).
2. The different basis vectors $\mathbf{e}_{(\mu)}$ do not necessarily have unit length.
3. The different basis vectors $\mathbf{e}_{(\mu)}$ may have different direction and magnitude at different spacetime points [*very important!*].

[**Relationship to the metric**] For all coordinate bases $\{\mathbf{e}_{(\mu)}\}$, the vector line element $d\boldsymbol{s}$ is always given by

$$d\boldsymbol{s} = dx^\mu \mathbf{e}_{(\mu)}, \tag{3}$$

without any extra factors on the right-hand side. This immediately leads to a relationship between the coordinate basis vectors and the metric

$$ds^2 = d\boldsymbol{s} \cdot d\boldsymbol{s} = \mathbf{e}_{(\mu)} \cdot \mathbf{e}_{(\nu)} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu, \tag{4}$$

that is $g_{\mu\nu} = \mathbf{e}_{(\mu)} \cdot \mathbf{e}_{(\nu)}$.

II. TRANSFORMATION OF COORDINATES AND CHANGE OF BASIS

You are already familiar with the concept of change of variables or coordinates in the context of basic integration. For example, let's take a coordinate system given by (u, v). Now, imagine that I want to define a new coordinate system (p, q) such that $p = p(u, v)$ and $q = q(u, v)$. Using the chain rule, I have

$$dp = \frac{\partial p}{\partial u} du + \frac{\partial p}{\partial v} dv, \qquad dq = \frac{\partial q}{\partial u} du + \frac{\partial q}{\partial v} dv, \tag{5}$$

or more compactly

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \tag{6}$$

where the prime coordinates correspond to (p, q) and the unprimed to (u, v). The object $\frac{\partial x'^\mu}{\partial x^\nu}$ is something you've seen before: it is the **Jacobian matrix** taking care of the change of coordinates.

[**Change to basis vectors**] Let's see how the change of coordinate above affect the corresponding coordinate basis used to write down components of vectors and tensors at any spacetime point. Consider the vector line element $d\boldsymbol{s}$. I'm free to write this vector in any coordinate basis I want, including that with basis vectors $\{\mathbf{e}_{(\mu)}\}$ or $\{\mathbf{e}'_{(\mu)}\}$

$$d\boldsymbol{s} = dx^\mu \mathbf{e}_{(\mu)} = dx'^\mu \mathbf{e}'_{(\mu)} = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \mathbf{e}'_{(\mu)}, \tag{7}$$

where we used Eq. (6) in the last step. Now, for the first equality to be equal to the last, we must have

$$\mathbf{e}_{(\nu)} = \frac{\partial x'^\mu}{\partial x^\nu} \mathbf{e}'_{(\mu)}, \tag{8}$$

which gives the transformation property of the basis vectors in a coordinate basis.

[**How vector components transform**] From this, it is straightforward to prove that the component of a vector $\boldsymbol{p}$ transform as

$$\boldsymbol{p} = p^\mu \mathbf{e}_{(\mu)} = p'^\mu \mathbf{e}'_{(\mu)} \qquad \rightarrow \qquad p'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} p^\nu. \tag{9}$$

This last equality is the *definition* of a vector: It is any object whose components transform in this fashion when changing coordinates.

[**How dual vector components transform**] The components of a dual vector (one-form) transform the opposite way

$$p'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} p_\nu \tag{10}$$

[**How the metric transforms**] We can repeat the same procedure for an object with two indices such as the metric (see **box 5.2**).

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}. \tag{11}$$

[**Key identity**] The following identity can be extremely useful when performing these coordinate transformations

$$\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\nu} = \frac{\partial x'^\mu}{\partial x'^\nu} = \delta^\mu{}_\nu. \tag{12}$$