PHYS 480/581: General Relativity
A Metric for the Universe

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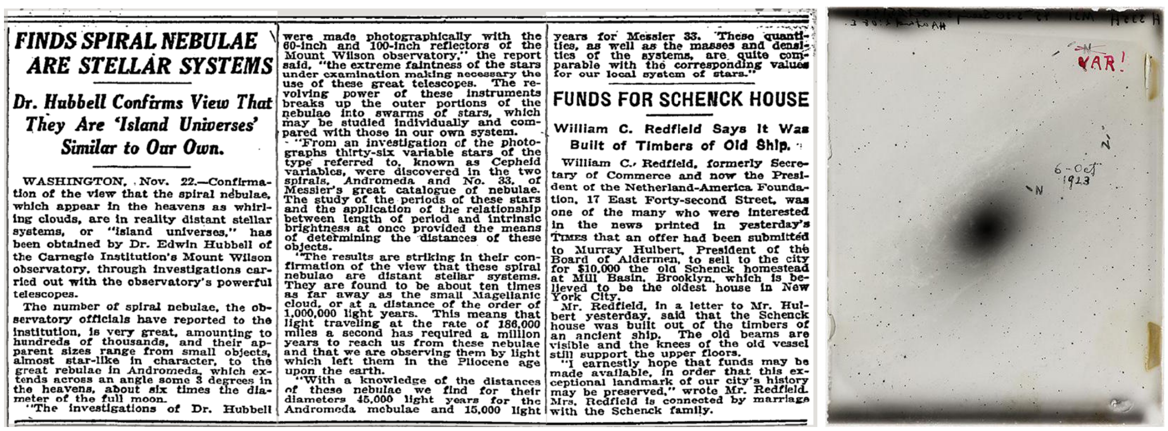
I. THE VERY LARGE, EXPANDING UNIVERSE

For a very long time, most astronomers thought that the whole Universe was simply our own galaxy. From there, some of the main steps towards establishing the very large, expanding Universe that we know today are:

- In 1924, when Hubble made a convincing argument that "nebulae" (fuzzy objects in the night sky, see right panel of Fig. 1) where not part of the Milky Way, but distinct galaxies. This was initially published in the New York Times on November 23rd, 1924 (see left panel of Fig. 1).



FINDS SPIRAL NEBULAE ARE STELLAR SYSTEMS

Dr. Hubbell Confirms View That They Are 'Island Universes' Similar to Our Own.

WASHINGTON, Nov. 22.—Confirmation of the view that the spiral nebulae, which appear in the heavens as whirling clouds, are in reality distant stellar systems, or "island universes," has been obtained by Dr. Edwin Hubbell of the Carnegie Institution's Mount Wilson observatory, through investigations carried out with the observatory's powerful telescopes.

The number of spiral nebulae, the observatory officials have reported to the institution, is very great, amounting to hundreds of thousands, and their apparent sizes range from small objects, almost star-like in character, to the great rebulae in Andromeda, which extends across an angle some 3 degrees in the heavens, about six times the diameter of the full moon.

"The investigations of Dr. Hubbell were made photographically with the 60-inch and 100-inch reflectors of the Mount Wilson observatory," the report said, "the extreme faintness of the stars under examination making necessary the use of these great telescopes. The revolving power of these instruments breaks up the outer portions of the nebulae into swarms of stars, which may be studied individually and compared with those in our own system.

"From an investigation of the photographs thirty-six variable stars of the type referred to, known as Cepheid variables, were discovered in the two spirals, Andromeda and No. 33, of Messier's great catalogue of nebulae. The study of the periods of these stars and the application of the relationship between length of period and intrinsic brightness at once provided the means of determining the distances of these objects.

"The results are striking in their confirmation of the view that these spiral nebulae are distant stellar systems. They are found to be about ten times as far away as the small Magellanic cloud, or at a distance of the order of 1,000,000 light years. This means that light traveling at the rate of 186,000 miles a second has required a million years to reach us from these nebulae and that we are observing them by light which left them in the Pliocene age upon the earth.

"With a knowledge of the distances of these nebulae we find for their diameters 45,000 light years for the Andromeda mebulae and 15,000 light years for Messier 33. These quantities, as well as the masses and densities of the systems, are quite comparable with the corresponding values for our local system of stars."

FUNDS FOR SCHENCK HOUSE

William C. Redfield Says It Was Built of Timbers of Old Ship.

William C. Redfield, formerly Secretary of Commerce and now the President of the Netherland-America Foundation, 17 East Forty-second Street, was one of the many who were interested in the news printed in yesterday's TIMES that an offer had been submitted to Murray Hulbert, President of the Board of Aldermen, to sell to the city for $10,000 the old Schenck homestead at Mill Basin, Brooklyn, which is believed to be the oldest house in New York City.

Mr. Redfield, in a letter to Mr. Hulbert yesterday, said that the Schenck house was built out of the timbers of an ancient ship. The old beams are visible and the knees of the old vessel still support the upper floors.

"I earnestly hope that funds may be made available, in order that this exceptional landmark of our city's history may be preserved," wrote Mr. Redfield. Mrs. Redfield is connected by marriage with the Schenck family.

FIG. 1. **Left Panel:** New York Times article announcing Hubble's discovery that nebulae are "island universes" (galaxies) similar to our own. **Right panel:** Photographic plate image of the Andromeda galaxy taken with the 100-inch Hooker telescope on Mount Wilson in 1923. Note the two stars labeled with N, marking them as variable stars (cepheids), which Hubble used to measure the distance to Andromeda.

- Then, Lemaître (in 1927) and Hubble (in 1929) established that there is a linear relationship between the distance d to a galaxies and its recession speed v

$$v \approx H_0 d, \tag{1}$$

where H_0 is called the Hubble constant. This is now known as the Hubble-Lemaître law. This linear relationship is illustrated in Fig. 2. This discovery showed that the Universe is expanding, a prediction of General Relativity initially rejected by Einstein, but then quickly accepted by the whole astronomical community. Note that Eq. (1) is only valid for relatively short distances, and need to be generalized for more distant galaxies.

II. THE COSMOLOGICAL PRINCIPLE

In the early days of General Relativity, Einstein postulated that a most natural universe would be both *homegeneous* and *isotropic*. This is now known as the cosmological principle.

- Isotropy: On average, the Universe looks the same in every direction; there is no special direction.
- Homogeneity: On average, every large-enough region looks the same; there is no preferred center or point in the Universe.

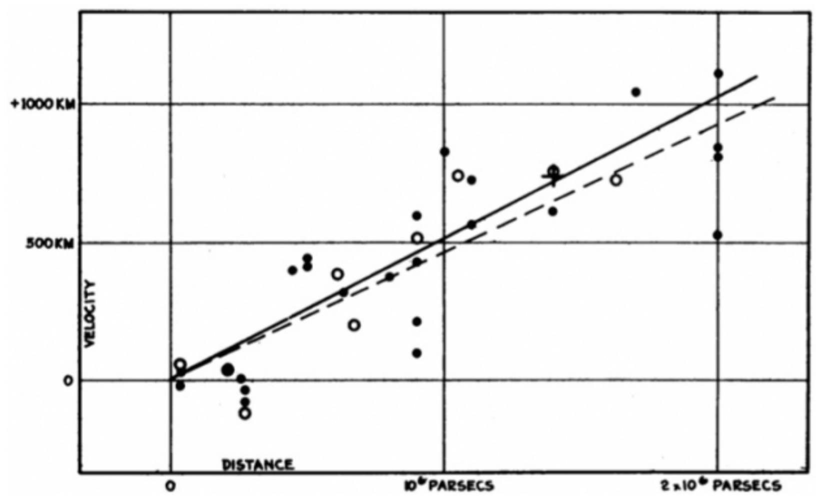


FIG. 2. First measurement of the Hubble-Lemaître law by Hubble in 1929. Note that errors in the measurement of distance lead Hubble to infer a value of H_0 that is nearly 10 times larger than the actual value known today.

Detailed observations of the cosmic microwave background (CMB) and of the distribution of galaxies on large scales validate the Cosmological Principle to a high degree of accuracy. The Cosmological Principle coupled with the expansion of the Universe means that the expansion has no preferred center. That is, just like we see every galaxy in the sky moving away from us, any observer in any of these other galaxies will also see every galaxies moving away from them. This is illustrated in Fig. 3 below, where you can think of galaxies on a regular cartesian grid. In an homogeneous isotropic expanding Universe, the grid itself is expanding uniformly: every galaxy see its nearest neighbors receding away.

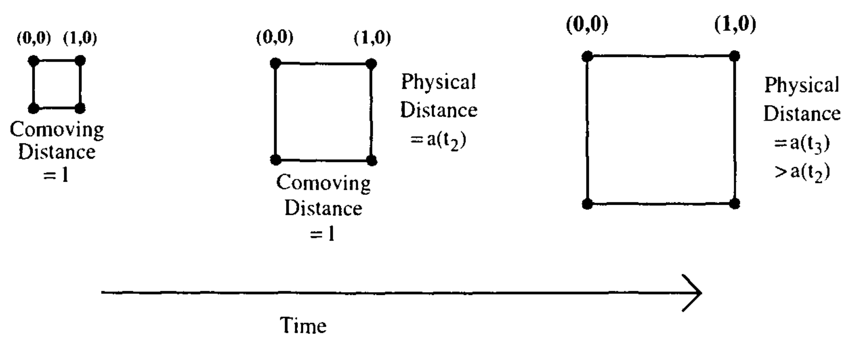


FIG. 3. Homogeneous and isotropic expansion. In comoving coordinates, the coordinates of the points are fixed, but the physical distance between them expands with the scale factor $a(t)$. Figure from Dodelson (2003).

III. THE METRIC OF THE UNIVERSE

We would like to write down a trial metric that respects the Cosmological Principle and can describe an expanding universe. In homework 9, we've already introduced the metric

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2], \tag{2}$$

where $a(t)$ is a dimensionless function of coordinate time called the *scale factor*. Note that this metric satisfies the Cosmological Principle: the coordinate x, y, and z are all treated equally so the metric is spatially isotropic, and none of the metric component depends explicitly on the coordinates (x, y, z) so the metric is the same at every point in space (that is, it is spatially homogeneous). Further, if $a(t)$ is an increasing function of time than it could describe the expansion of the Universe. Thus, the metric above is a good first guess. The coordinates used to write it down are called *comoving* coordinates. In this coordinate system, the spatial coordinates of events are fixed, but the coordinate grid itself is expanding (see Fig. 3).

A. Distances and Ages

On the coordinate grid, the comoving distance $d_{\rm c}$ between coordinate points is always fixed. To compute the *physical* distance $d_{\rm p}$ between two grid points, once simply needs to multiply the comoving distance by the value of the scale factor at the time of interest, that is,

$$d_{\rm p}(t) = a(t) d_{\rm c}. \tag{3}$$

Note that we usually normalize $a(t)$ such that $a(t_0) = 1$ today, where t_0 is the age of the Universe. So, *today* the physical and comoving distance between two spacetime events coincide, but this wasn't true in the past and won't be true in the future. It is also usually assumed that at the Big Bang ($t = 0$), we have $a(0) = 0$.

An example of this is the comoving distance travelled by a streaming photon moving along the x-axis since the Big Bang ($t = 0$). Since photons travel on null (lightlike) trajectories, we have $ds^2 = 0$. The comoving distance is then

$$\begin{aligned} dx &= \frac{dt}{a(t)} \\ \int_0^{d_{\rm c}} dx &= \int_0^t \frac{dt'}{a(t')} \\ d_{\rm c}(t) &= \int_0^t \frac{dt'}{a(t')}. \end{aligned} \tag{4}$$

The *physical* distance travelled by the photon is then

$$d_{\rm p}(t) = a(t) \int_0^t \frac{dt'}{a(t')}. \tag{5}$$

Note that to solve these integrals, we need to know the behavior of $a(t)$, which is gotten by solving the Einstein equation. At this point, it is important to introduce the *Hubble rate*

$$H \equiv \frac{\dot{a}}{a}, \tag{6}$$

where a dot represents a derivative with respect to coordinate time t. In general, the Hubble rate H is a function of time. Again, this time dependence will be specified via the Einstein equation. The Hubble rate provides us a way to compute the age of the Universe (and more generally, the amount of coordinate time between any two spacetime events). Using the fact that

$$\begin{aligned} \frac{da}{dt} &= aH \\ \frac{da}{aH} &= dt, \end{aligned} \tag{7}$$

the age of the Universe is

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_0^1 \frac{da}{aH}. \end{aligned} \tag{8}$$

B. Spatial Curvature

However, Eq. (2) is not the only metric respecting all the symmetries required by the Cosmological Principle and the expansion. In the above metric, two lines that are initially parallel will stay parallel forever. But note that none of the requirement for the symmetries of the cosmological metric enforces that. So we are free to consider spatial metrics for which two initially parallel lines either converge or diverge. Thus, in addition to the *spatially flat* case given in Eq. (2), there are two other spatial geometries that we can have

- A spatial geometry where two initially parallel lines converge looks like a sphere. This is referred to as a *closed* universe.
- A spatial geometry where two initially parallel lines diverge looks like a saddle. This is referred to as a *open* universe.

Again, whether the Universe is flat, open, or closed is not an arbitrary choice; it is dictated by the Einstein equation depending on the energy-momentum content of the Universe.

For non-flat spatial slices, it is easiest to write down the metric in terms of spherical-like comoving coordinates

$$ds^2 = -dt^2 + a^2(t)[r^2 + q^2(r)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{9}$$

where r is the comoving radial distance from the origin. The function $q(r)$ enforces the spatial geometry; for a spatially flat universe we must have $q(r) = r$. Determining the behavior of $a(t)$ and $q(r)$ requires us to write down the Einstein equation. These steps are outlined in Moore and we won't reproduce them here:

$$q(r) = \begin{cases} R\sin(r/R) & \text{if } \kappa > 0, \text{ (closed)} \\ r & \text{if } \kappa = 0, \text{ (flat)} \\ R\sinh(r/R) & \text{if } \kappa < 0, \text{ (open)} \end{cases} \tag{10}$$

where $\kappa = \pm 1/R^2$, where R is the radius of curvature of spatial slices. These expressions make clear that only for $r \sim R$ will we start being sensitive to the presence of spatial curvature, since for $r/R \ll 1$, $R\sin(r/R) \sim R\sinh(r/R) \sim r$. The sign and value of κ will be dictated by the Einstein equation. We will look at this next, but before doing so let me note that observations indicate that our Universe is *spatially flat* to very good accuracy. Thus, while the discussion of spatial curvature is interesting at the physical level, it doesn't seem to be very relevant for the structure of our Universe.