PHYS 480/581: General Relativity
Cosmological Evolution

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I. EVOLUTION OF THE SCALE FACTOR

Last time, we saw that the metric describing the geometry of the Universe

$$ds^2 = -dt^2 + a^2(t)[r^2 + q^2(r)(d\theta^2 + \sin^2\theta d\phi^2)], \tag{1}$$

where r is the comoving radial distance from the origin of the coordinate system. The function $q(r)$ enforces the spatial geometry and is given by

$$q(r) = \begin{cases} R\sin(r/R) & \text{if } \kappa > 0, \text{ (closed)} \\ r & \text{if } \kappa = 0, \text{ (flat)} \\ R\sinh(r/R) & \text{if } \kappa < 0, \text{ (open)} \end{cases} \tag{2}$$

where $\kappa = \pm 1/R^2$, where R is the radius of curvature of spatial slices. Determining the behavior of $a(t)$ and $q(r)$ requires us to write down the Einstein equation.

A. The Einstein Equation

In homework 9, you derived the Einstein equation in the spatially flat case ($q(r) = r$). The derivation with curvature follows the same trajectory and yields

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho_{\rm tot} - \frac{\kappa}{a^2}, \tag{3}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho_{\rm tot} + 3p_{\rm tot}), \tag{4}$$

which are the two Friedmann equations. Note that the curvature term enters only in the first equation. Here, we have taken the stress-energy tensor to be

$$T^{\mu}{}_{\nu} = \text{diag}(-\rho_{\rm tot}, p_{\rm tot}, p_{\rm tot}, p_{\rm tot}). \tag{5}$$

where

$$\rho_{\rm tot} = \sum_i \rho_i, \tag{6}$$

where the sum runs over all the constituents of the Universe, including the cosmological constant (dark energy). We will go over these constituents in the next section. But, first, let's note that that Eqs. (3) and (4) do not completely determine the evolution of of $a(t)$, $\rho_{\rm tot}$ and $p_{\rm tot}$ (2 equations for 3 unknowns). To close this system of equation, we need to specify an *equation of state* w_i linking the pressure p_i of every constituent of the Universe to its energy density ρ_i

$$p_i = w_i\rho_i. \tag{7}$$

Equipped with the equation of state, we can solve for the evolution of the energy density ρ and the scale factor $a(t)$. In homework 9, you used the conservation of the stress-energy tensor $\nabla_\mu T^{\mu}{}_{\nu} = 0$. Since this equation is a consequence of the Einstein, it does not contain any information that was not already contained in the Einstein equation. Let's see this by computing the evolution of the energy density from Eqs. (3) and (4). First, let's multiply by a^2 both sides of Eq. (3) and take the time derivative

$$\frac{d}{dt}\left(\dot{a}^2\right) = \frac{d}{dt}\left(\frac{8\pi G}{3}a^2\rho - \kappa\right)$$
$$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(a^2\dot{\rho} + 2a\dot{a}\rho). \tag{8}$$

Now, divide both sides by $2a^2$ and use Eq. (4) to substitute $\ddot{a}/a$

$$\frac{\dot{a}}{a}\frac{\ddot{a}}{a} = \frac{4\pi G}{3}(\dot{\rho} + 2\frac{\dot{a}}{a}\rho)$$
$$-\frac{\dot{a}}{a}\frac{4\pi G}{3}(\rho + 3p) = \frac{4\pi G}{3}(\dot{\rho} + 2\frac{\dot{a}}{a}\rho)$$
$$0 = \dot{\rho} + 2\frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}(\rho + 3p)$$
$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p)$$
$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w). \tag{9}$$

Note that the curvature is irrelevant to the cosmological evolution of ρ, reflecting the local nature of energy conservation. Let's solve this equation for a constant w. First note that the above equation can be written as

$$\frac{1}{\rho a^3}\frac{d}{dt}\left(\rho a^3\right) = \frac{-3w}{a}\frac{da}{dt}. \tag{10}$$

Setting $u \equiv \rho a^3$, we get

$$\frac{1}{u}du = \frac{-3w}{a}da$$
$$\int \frac{1}{u}du = \int \frac{-3w}{a}da$$
$$\ln u = -3w \ln a + C$$
$$\ln u = \ln a^{-3w} + C$$
$$u = Ka^{-3w}$$
$$\rho a^3 = Ka^{-3w}$$
$$\rho = Ka^{-3(1+w)}. \tag{11}$$

B. Constituents and their cosmological evolution

Our Universe appears to contain, in order from smallest to largest contribution to the total energy density:

1. **Neutrinos:** Our Universe is permeated with a cosmological neutrino background, a relic from the epoch of Nucleosynthesis in the first few seconds of our Universe. These neutrinos are slightly colder than the CMB photons (see next) and were relativistic for most of the history of the Universe with $w = 1/3$.

2. **Photons:** Our Universe is also permeated with a cosmological background of photons which were emitted during the epoch of recombination when the Universe became neutral. Since photons are massless, they always behave like radiation with $w = 1/3$.

3. **Baryons:** The "normal" matter that makes stars, planets, giant gas clouds, dust, etc. By far, most of the baryons in the Universe are in cold gas clouds made of hydrogen and helium. Baryons form about 5% of all energy density of the Universe. On average, baryons are highly non-relativistic particles and thus have $w \simeq 0$.

4. **Dark Matter:** A form of non-baryonic matter that is needed to explain many observations (structure of galaxies, galaxy clusters, the anisotropies of the cosmic microwave background, etc). It appears to be highly non-relativistic (such that it can form gravitationally bound structure) with $w = 0$. It makes up about 26% of the total energy density of the Universe.

5. **Dark Energy:** It appears that our Universe is dominated by something that we call "dark energy". As far as we can tell, it's properties seem to be very close to that of a cosmological constant Λ, with an equation of state $w = -1$. It makes up about 69% of the total energy density of the Universe today.

We can use Eq. (11) to get the cosmological evolution of these different components: For dark matter and baryons (for which $w = 0$), we have $\rho \propto a^{-3}$, while for radiation (massless neutrinos and photons) for which $w = 1/3$, we have $\rho \propto a^{-4}$. For dark energy ($w = -1$), we have $\rho_\Lambda = a^0 = \text{const}$.

C. Scale factor evolution

Equipped with these solutions, we can go back to the first Friedmann equation (Eq. (3)) to compute the evolution of the scale factor itself. Let's remember that we normalize $a(t)$ such that $a(t_0) = 1$, such that we can write

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\frac{\rho_{m0}}{a^3} + \frac{\rho_{r0}}{a^4} + \rho_\Lambda\right) - \frac{\kappa}{a^2}, \tag{12}$$

where ρ_{m0} and ρ_{r0} are the energy density in matter (baryon and dark matter) and radiation (photons and neutrinos) today, respectively. We can introduce the *critical energy density of the Universe*

$$\rho_\mathrm{c} = \frac{3H_0^2}{8\pi G}, \tag{13}$$

where H_0 is the Hubble rate today (sometime called the Hubble constant). Substituting this in the above yields

$$\begin{aligned} H^2 &= H_0^2\left(\frac{\rho_{m0}}{\rho_\mathrm{c} a^3} + \frac{\rho_{r0}}{\rho_\mathrm{c} a^4} + \frac{\rho_\Lambda}{\rho_\mathrm{c}}\right) - \frac{\kappa}{a^2} \\ &= H_0^2\left(\frac{\Omega_\mathrm{m}}{a^3} + \frac{\Omega_\mathrm{r}}{a^4} + \Omega_\Lambda + \frac{\Omega_K}{a^2}\right), \end{aligned} \tag{14}$$

where we have defined the dimensionless density parameters

$$\Omega_\mathrm{m} \equiv \frac{\rho_{m0}}{\rho_\mathrm{c}}, \qquad \Omega_\mathrm{r} \equiv \frac{\rho_{r0}}{\rho_\mathrm{c}}, \qquad \Omega_\Lambda \equiv \frac{\rho_\Lambda}{\rho_\mathrm{c}}, \tag{15}$$

and where we have *defined* the energy density in curvature as

$$\Omega_K \equiv -\frac{\kappa}{H_0^2}. \tag{16}$$

The critical density ρ_c plays a key role since it is the energy density that the Universe needs to have in order to be *spatially flat*. Indeed, evaluating Eq. (14) at the present time when $a = 1$ and $H = H_0$ yields

$$1 = \Omega_\mathrm{m} + \Omega_\mathrm{r} + \Omega_\Lambda + \Omega_K, \qquad \Rightarrow \qquad 1 - \Omega_\mathrm{m} - \Omega_\mathrm{r} - \Omega_\Lambda = \Omega_K, \tag{17}$$

and thus having a spatially-flat Universe ($\Omega_K = 0$) means that

$$\sum_i \Omega_i = 1, \qquad \Rightarrow \qquad \rho_\mathrm{tot} = \rho_\mathrm{c}. \tag{18}$$

In our Universe, we appear to have $\Omega_K \approx 0$, and thus we do have $\sum_i \Omega_i \approx 1$.

We can solve the Friedmann equation for a single-constituent Universe with density parameter Ω. In this case the equation takes the form

$$\frac{\dot{a}}{a} = H_0\left(\frac{\Omega}{a^n}\right)^{1/2}. \tag{19}$$

This can be solved by separation of variables

$$\begin{aligned} \int a^{\frac{n}{2}-1} da &= \int H_0\sqrt{\Omega} dt \\ \frac{2}{n} a^{n/2} &= H_0\sqrt{\Omega} t + C \\ a(t) &= \left(\frac{n H_0 \sqrt{\Omega} t}{2} + C\right)^{2/n}. \end{aligned} \tag{20}$$

Demanding that $a(0) = 0$ yields $C = 0$. We also recognize that

$$\frac{2}{nH_0\sqrt{\Omega}} = t_0 \tag{21}$$

is the age of the Universe in such a cosmology since

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_0^1 \frac{da}{aH} \\ &= \frac{1}{H_0\sqrt{\Omega}} \int_0^1 a^{\frac{n}{2}-1} da \\ &= \frac{1}{H_0\sqrt{\Omega}} \frac{2}{n} a^{n/2}\Big|_0^1 \\ &= \frac{2}{nH_0\sqrt{\Omega}}. \end{aligned} \tag{22}$$

We thus can write the solution to the Friedmann equation for a single-component universe as

$$a(t) = \left(\frac{t}{t_0}\right)^{2/n}. \tag{23}$$

This single-component solution of course does *not* describe our Universe (which has multiple components), but it is useful to describe the evolution of the Universe when one component *dominates* to the energy density of the Universe. For instance, at early times, the Universe is radiation dominated ($n = 4$), implying

$$a(t) \propto t^{1/2} \qquad \text{(radiation domination)}. \tag{24}$$

Once matter ($n = 3$) becomes more important at later times, we then have

$$a(t) \propto t^{2/3} \qquad \text{(matter domination)}. \tag{25}$$

For curvature domination ($n = 2$), we have

$$a(t) \propto t \qquad \text{(curvature domination)}. \tag{26}$$

For dark energy domination ($n = 0$), we have to back to the integral to determine the result. This leads to

$$a(t) \propto e^{H_0 t} \qquad \text{(dark energy domination)}, \tag{27}$$

where H_0 is the Hubble rate, which becomes a constant in this case.