PHYS 480/581: General Relativity
Hawking Radiation and Black Hole Thermodynamics

(Dated: April 12, 2024)

I. SPONTANEOUS PARTICLE CREATION NEAR THE EVENT HORIZON

Positive and Negative Energy When discussing the geodesic equation for the motion of test particle in the Schwarzschild geometry, we saw that the relativistic energy per unit mass

$$e = \left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}, \tag{1}$$

is always conserved. Now, for $r > 2GM$, this energy is always positive as t is timelike in this region. For $r < 2GM$ however, $(1 - 2GM/r) < 0$ and $dt/d\tau$ can have either sign since t is spacelike in this region. Thus, inside the event horizon, we can have particles with *negative* relativistic energy per unit mass. This might seem a bit academic at this point as we know that anything within the event horizon will eventually reach the singularity at $r = 0$, and thus have no impact on a far-away observer.

Impact of Quantum Mechanics Nevertheless, once quantum mechanics (really, quantum field theory) is taken into account, the fact that negative energy states are allowed by General Relativity becomes important. Indeed, quantum field theory allows particle-antiparticle pairs to pop out of the vacuum. In flat spacetime these particle-antiparticle pairs only exist for a tiny amount of time before annihilating back to the vacuum. Their possible duration Δt is approximately given by the Heisenberg uncertainty principle $\Delta t \sim \hbar/E$, where E is the energy of one of the component of the particle-antiparticle pair.

Pair creation at the horizon Now, what happens is this spontaneous creation of particle-antiparticle pairs occurs near the event horizon of a black hole? Since energy is conserved, we can say that one particle will have energy E, while the other particle has energy $-E$. Imagine that the negative energy particle falls through the event horizon. Since negative energy state are *classically* allowed by General Relativity, there is no longer the need for this particle to annihilate in the short time $\Delta t \sim \hbar/E$ and this particle with $E < 0$ can follow its geodesic all the way to $r = 0$, effectively *reducing* the mass of the black hole. By momentum conservation, the other particle from the pair (that with $E > 0$) will travel away from the black hole. From the point of view of an observer at infinity, it looks like the black hole is *radiating* energy away. This radiation is called *Hawking radiation*.

II. ENERGY OF OUTGOING PARTICLES AND HAWKING TEMPERATURE

Let's try to estimate the energy of the radiated particles and how it is related to the mass M of the black hole. Consider a particle-antiparticle pair popping out of the vacuum at rest at a radial coordinate $r = 2GM + \epsilon$, with $\epsilon \ll 2GM$. If the negative energy particle is to cross into the event horizon, it has to survive for an amount of proper time

$$\Delta\tau \sim \frac{\hbar}{E}. \tag{2}$$

The particle will follow a geodesic as it makes its way into the event horizon. From the normalization of the four-velocity $g_{\mu\nu}u^\mu u^\nu = -1$, we can derive the following equation

$$\frac{dr}{d\tau} = \pm\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)}. \tag{3}$$

Since the particle is initially at rest, we have $dr/d\tau = 0$ and $\ell = 0$ initially, which implies that

$$0 = e^2 - \left(1 - \frac{2GM}{2GM + \epsilon}\right)\left(1 + \frac{0}{r^2}\right) \quad \Rightarrow \quad e^2 = 1 - \frac{2GM}{2GM + \epsilon}. \tag{4}$$

The subsequent evolution of the particle is gotten by isolating $d\tau$ in Eq. (3) and integrating on both sides

$$\Delta\tau = -\int_{2GM+\epsilon}^{2GM} dr \frac{1}{\sqrt{-\frac{2GM}{2GM+\epsilon} + \frac{2GM}{r}}} \tag{5}$$

Now, make the substitution $\rho = r - 2GM$, $d\rho = dr$.

$$\begin{aligned}
\Delta\tau &= -\int_{2GM+\epsilon}^{2GM} dr \frac{1}{\sqrt{-\frac{2GM}{2GM+\epsilon} + \frac{2GM}{r}}} \\
&= \int_0^\epsilon d\rho \frac{1}{\sqrt{-\frac{2GM}{2GM+\epsilon} + \frac{2GM}{\rho+2GM}}} \\
&= \int_0^\epsilon d\rho \frac{1}{\sqrt{-\frac{1}{1+(\epsilon/2GM)} + \frac{1}{1+\rho/(2GM)}}} \\
&\approx \int_0^\epsilon d\rho \frac{1}{\sqrt{-(1-\frac{\epsilon}{2GM}) + 1 - \frac{\rho}{2GM}}} \\
&\approx \int_0^\epsilon d\rho \frac{1}{\sqrt{\frac{\epsilon}{2GM} - \frac{\rho}{2GM}}} \\
&\approx \int_0^\epsilon d\rho \frac{\sqrt{2GM}}{\sqrt{\epsilon-\rho}} \\
&\approx \sqrt{2GM}(-2\sqrt{\epsilon-\rho})\Big|_0^\epsilon \\
&\approx 2\sqrt{2GM\epsilon}.
\end{aligned} \tag{6}$$

Thus the characteristic energy of the particle will be

$$E \sim \frac{\hbar}{\Delta\tau} = \frac{\hbar}{2\sqrt{2GM\epsilon}}. \tag{7}$$

Now, we would like to convert this value of the energy to what an observer at infinity will see. In **box 16.2**, we derive that

$$E_\infty = \sqrt{1 - \frac{2GM}{r}} E. \tag{8}$$

Plugging $r = 2GM + \epsilon$ and the value of E from Eq. (7), we then get

$$E_\infty = \sqrt{1 - \frac{2GM}{2GM+\epsilon}} \frac{\hbar}{2\sqrt{2GM\epsilon}} = \sqrt{\frac{\epsilon}{2GM}} \frac{\hbar}{2\sqrt{2GM\epsilon}} \approx \frac{\hbar}{4GM}, \tag{9}$$

which is independent of ϵ.

Very small energy for large black holes Note that $E_\infty \propto 1/M$, which means that particles coming out of a more massive black hole will have a *lower* energy than those coming out from a smaller mass black hole. Also, it's important to realize that this energy is *tiny* for typical astrophysical black holes. This means that only photons (which are massless) can really be emitted by this mechanism, as other known particles (electrons, neutrinos, etc.) are too massive to be created in this fashion. Importantly, note that the factor of $1/(4GM)$ is the local gravitational acceleration at the event horizon. Indeed, just like the gravitational acceleration at the surface of the Earth is

$$g = \frac{GM_\oplus}{R_\oplus^2}, \tag{10}$$

the gravitational acceleration at the event horizon will be

$$\kappa = \frac{GM}{r_s^2} = \frac{GM}{(2GM)^2} = \frac{1}{4GM}. \tag{11}$$

Here, the terminology of *surface gravity* is usually used when referring to κ.

Thermal radiation out of a black hole In general, we expect the radiation coming out the black hole to have an energy distribution centered on the typical energy given in Eq. (9). In fact, Hawking showed that the energy spectrum of the outgoing radiation is exactly that of a blackbody with temperature given by

$$k_B T_H = \frac{\hbar\kappa}{2\pi}, \tag{12}$$

where k_B is Boltzmann's constant. Plugging in some numbers yields

$$T_H = \frac{6.17 \times 10^{-8}\mathrm{K}}{M/M_\odot}, \tag{13}$$

which implies that a solar mass black hole emits radiation with a temperature $T \sim 60$ nK $\ll T_{\mathrm{CMB}} = 2.725$ K. This radiation is nearly impossible to detect against such a hotter foreground of cosmic microwave background photons.

Surface gravity and energy emitted We note that the surface gravity for a non-rotating black hole (including the case where the black hole has an electric charge), the surface gravity is given by

$$\kappa = \frac{1}{2}\frac{\partial}{\partial r}(g_{tt})\Big|_{r=r_{\mathrm{EH}}}, \tag{14}$$

where r_{EH} is the radius of the event horizon. Since the hawking temperature is proportional to the surface gravity, one way to interpret a high Hawking temperature is that it is coming from a given region of *high* spacetime curvature. Basically, since it takes a lot of mass-energy to curve spacetime significantly, regions of high curvature have a lot of energy to radiate and thus shine hotter in Hawking radiation. We can think of these highly curved regions of spacetime as tightly-compressed springs, with Hawking radiation providing a mechanism to slowly unwind the tension in the spring and relax spacetime to its natural state, flat spacetime.

Lifetime of a black hole The fact that black holes can radiate energy away implies that they can *evaporate* over time. Using the Stefan-Boltzmann law, the rate of mass loss for a black hole is given by

$$\frac{dM}{dt} = -A\sigma T_H^4, \tag{15}$$

where A is the area of the event horizon and σ is the Stefan-Boltzmann constant. This can be integrated to give the lifetime of the black hole (**box 16.4**)

$$\tau_{\mathrm{life}} = \frac{256\pi^3 k_B^4}{3G\sigma\hbar^4}(GM)^3. \tag{16}$$

For astrophysical black holes, this lifetime is very long, much longer than the age of the Universe.

III. BLACK HOLE ENTROPY

Thermodynamics implies that any object that has a temperature must have an accompanying entropy S, via the relation

$$\frac{1}{T} = \frac{\partial S}{\partial U}, \tag{17}$$

where U is the internal energy, which for a black hole is simply its mass. We can integrate the above expression to obtain an expression for the entropy of a black hole

$$S = \frac{k_B A}{4G\hbar}, \tag{18}$$

where A is the area of the event horizon. So, the entropy of a black hole is proportional to its *area*. This is weird, as usually entropy scales with the *volume* of an object. Since the entropy is given by $S = k_b \ln\Omega$, where Ω is the number of microstates that are internal to the black holes. What are there microstates? We don't know for sure, but this is something that can only be understood from the point of view of quantum gravity, for which we don't yet have a complete theory. All we can say now is that any successful theory of quantum theory of quantum gravity will have to explain the area law for black hole entropy.