PHYS 480/581: General Relativity
Levi-Civita Symbol

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I. THE COMPLETELY ANTI-SYMMETRIC SYMBOL

Consider the following completely antisymmetric object

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} = \begin{cases} +1 & \text{if}\, \mu\nu\sigma\rho\, \text{are even permutation of}\, 0123, \\ -1 & \text{if}\, \mu\nu\sigma\rho\, \text{are odd permutation of}\, 0123, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The main question we want to answer here is whether $\tilde{\epsilon}_{\mu\nu\sigma\rho}$ is a tensor. Consider the combination

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} \frac{\partial x^\rho}{\partial x'^\delta} = \tilde{\epsilon}_{\mu\nu\sigma\rho} J^\mu_\alpha J^\nu_\beta J^\sigma_\gamma J^\rho_\delta, \tag{2}$$

where we have used the notation

$$J^\mu_\alpha \equiv \frac{\partial x^\mu}{\partial x'^\alpha}. \tag{3}$$

To obtain some intuition about what the above combination is equal to, consider a two-dimensional example with

$$J^\mu_\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{4}$$

We then have

$$\tilde{\epsilon}_{\mu\nu} J^\mu_\alpha J^\nu_\beta = J^0_\alpha J^1_\beta - J^1_\alpha J^0_\beta. \tag{5}$$

These are for equations, one for each choice of α, β. Let's tabulate the results for each possible choice: These results

α	β	$\tilde{\epsilon}_{\mu\nu} J^\mu_\alpha J^\nu_\beta$
0	0	$ac - ca = 0$
0	1	$ad - cb = \det J$
1	0	$bc - da = -\det J$
1	1	$bd - db = 0$

can be summarized via the simple result

$$\tilde{\epsilon}_{\mu\nu} J^\mu_\alpha J^\nu_\beta = \tilde{\epsilon}'_{\alpha\beta} \det J. \tag{6}$$

The same results apply in four dimensions

$$\tilde{\epsilon}_{\mu\nu\sigma\rho} J^\mu_\alpha J^\nu_\beta J^\sigma_\gamma J^\rho_\delta = \tilde{\epsilon}'_{\alpha\beta\gamma\delta} \det J = \tilde{\epsilon}'_{\alpha\beta\gamma\delta} \left| \frac{\partial x}{\partial x'} \right|, \tag{7}$$

where we have used the common notation $\left| \frac{\partial x}{\partial x'} \right|$ to denote the determinant of the Jacobian for the coordinate transformation $x^\mu \to x'^\mu$. We can invert the above relation to write

$$\tilde{\epsilon}'_{\alpha\beta\gamma\delta} = \left| \frac{\partial x'}{\partial x} \right| \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \frac{\partial x^\sigma}{\partial x'^\gamma} \frac{\partial x^\rho}{\partial x'^\delta} \tilde{\epsilon}_{\mu\nu\sigma\rho}, \tag{8}$$

where we have used the fact that $\det(J^{-1}) = (\det J)^{-1}$. The above is not the transformation law of tensors due to the presence of the determinant of the Jacobian. Thus, $\tilde{\epsilon}_{\mu\nu\sigma\rho}$ is not a tensor, which is why we called it the Levi-Civita *symbol*. In fact, $\tilde{\epsilon}_{\mu\nu\sigma\rho}$ is an example of **tensor densities**.