PHYS 480/581
General Relativity

Review before midterm

Question 1.

Show that for a *diagonal* metric $g_{\mu\nu}$, we have

(a) $\Gamma^{\lambda}_{\mu\nu} = 0$,

(b) $\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu}$,

(c) $\Gamma^{\lambda}_{\mu\lambda} = \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right)$,

(d) $\Gamma^{\lambda}_{\lambda\lambda} = \partial_\lambda\left(\ln\sqrt{|g_{\lambda\lambda}|}\right)$,

where $\mu \neq \nu \neq \lambda$ and repeated indices are *not* summed over.
Solutions:
We first note that for a diagonal metric, we the inverse metric components are $g^{\lambda\lambda} = 1/g_{\lambda\lambda}$. For part (a),

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right) = 0, \tag{1}$$

since all three metric components appearing in the above are all off-diagonal (since $\mu \neq \nu \neq \lambda$) are thus zero. For part (b),

$$\begin{aligned}\Gamma^{\lambda}_{\mu\mu} &= \frac{1}{2}g^{\lambda\lambda}\left(\partial_\mu g_{\mu\lambda} + \partial_\mu g_{\lambda\mu} - \partial_\lambda g_{\mu\mu}\right)\\ &= -\frac{1}{2}g^{\lambda\lambda}\partial_\lambda g_{\mu\mu}\\ &= -\frac{1}{2g_{\lambda\lambda}}\partial_\lambda g_{\mu\mu},\end{aligned} \tag{2}$$

since $g_{\mu\lambda} = g_{\lambda\mu} = 0$ for $\mu \neq \lambda$. For part (c),

$$\begin{aligned}\Gamma^{\lambda}_{\mu\lambda} &= \frac{1}{2}g^{\lambda\lambda}\left(\partial_\mu g_{\lambda\lambda} + \partial_\lambda g_{\lambda\mu} - \partial_\lambda g_{\mu\lambda}\right)\\ &= \frac{1}{2g_{\lambda\lambda}}\partial_\mu g_{\lambda\lambda}\\ &= \frac{1}{2}\partial_\mu\left(\ln g_{\lambda\lambda}\right)\\ &= \partial_\mu\left(\ln\sqrt{|g_{\lambda\lambda}|}\right),\end{aligned} \tag{3}$$

where the absolute value is needed to ensure that resulting Christoffel is real. For part (d),

$$\begin{aligned}\Gamma^{\lambda}_{\lambda\lambda} &= \frac{1}{2} g^{\lambda\lambda} \left(\partial_\lambda g_{\lambda\lambda} + \partial_\lambda g_{\lambda\lambda} - \partial_\lambda g_{\lambda\lambda}\right) \\ &= \frac{1}{2g_{\lambda\lambda}} \partial_\lambda g_{\lambda\lambda} \\ &= \partial_\lambda \left(\ln \sqrt{|g_{\lambda\lambda}|}\right). \end{aligned} \tag{4}$$

Question 2.

Consider the metric

$$ds^2 = \frac{dp^2}{1 - kp^2} + p^2 dq^2, \tag{5}$$

where k is a real constant.

(a) Does this metric describe a curved or flat space?

(b) Is this space maximally symmetric?

(c) Write the p and q components of the geodesic equation.

Solutions:

To determine if a space is curved, we need to see if the Riemann tensor is non-vanishing. In two dimensions, there is only one independent component of the Riemann tensor, which we can take to be $R^p{}_{qpq}$. Let's first compute the Christoffel connection coefficients (in 2D we have 6 independent components; in general in n dimensions we have $n^2(n+1)/2$ independent Christoffels). Note that the metric is independent of the coordinate q, so all q derivatives vanish. Using the results from the above question, we have

$$\begin{aligned}\Gamma^p_{qq} &= -\frac{1}{2}(1 - kp^2)\partial_p g_{qq} \\ &= -\frac{1}{2}(1 - kp^2)\partial_p (p^2) \\ &= -p(1 - kp^2). \end{aligned} \tag{6}$$

We also have $\Gamma^q_{pp} = \Gamma^q_{qq} = \Gamma^p_{qp} = 0$ since they all involve q derivatives. We are only missing two connection coefficients

$$\begin{aligned}\Gamma^q_{pq} &= \partial_p \left(\ln \sqrt{g_{qq}}\right) \\ &= \partial_p \ln p \\ &= \frac{1}{p}, \end{aligned} \tag{7}$$

$$\begin{aligned}\Gamma^p_{pp} &= \frac{1}{2g_{pp}}\partial_p g_{pp} \\ &= \frac{1-kp^2}{2}\partial_p\left(\frac{1}{1-kp^2}\right) \\ &= -\frac{1-kp^2}{2}\frac{-2kp}{(1-kp^2)^2} \\ &= \frac{kp}{1-kp^2} \end{aligned} \tag{8}$$

The Riemann tensor is then

$$\begin{aligned} R^p{}_{qpq} &= \partial_p\Gamma^p_{qq} - \partial_q\Gamma^p_{pq} + \Gamma^p_{p\lambda}\Gamma^\lambda_{qq} - \Gamma^p_{q\lambda}\Gamma^\lambda_{qp} \\ &= \partial_p\left(-p(1-kp^2)\right) + \Gamma^p_{pp}\Gamma^p_{qq} - \Gamma^p_{qq}\Gamma^q_{qp} \\ &= -1+3kp^2 - \frac{kp}{1-kp^2}p(1-kp^2) + p(1-kp^2)\frac{1}{p} \\ &= kp^2. \end{aligned} \tag{9}$$

Since the Riemann is nonzero, then this space is curved. Let's compute the Ricci tensor and Ricci scalar.

$$\begin{aligned} R_{pp} &= R^\lambda{}_{p\lambda p} \\ &= R^q{}_{pqp} \\ &= g^{qq}g_{pp}R^p{}_{qpq} \\ &= \frac{1}{p^2(1-kp^2)}kp^2 \\ &= \frac{k}{1-kp^2}. \end{aligned} \tag{10}$$

$$\begin{aligned} R_{qq} &= R^\lambda{}_{q\lambda q} \\ &= R^p{}_{qpq} \\ &= kp^2. \end{aligned} \tag{11}$$

And finally $R_{pq} = R^\lambda{}_{q\lambda p} = 0$. The scalar curvature is then

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu} \\ &= g^{pp}R_{pp} + g^{qq}R_{qq} \\ &= (1-kp^2)\frac{k}{1-kp^2} + \frac{1}{p^2}kp^2 \\ &= 2k, \end{aligned} \tag{12}$$

which is a constant. Thus this represents a space that has the same curvature everywhere. This seems to suggest that this space is highly symmetric. For a maximally symmetric space, we must have

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right). \tag{13}$$

So for us,

$$\begin{aligned} R_{pqpq} &= \frac{2k}{2}\left(g_{pp}g_{qq} - g_{pq}g_{qp}\right) \\ &= k\frac{1}{1-kp^2}p^2 \\ &= \frac{kp^2}{1-kp^2}, \end{aligned} \tag{14}$$

which is consistent with what we have above.