PHYS 480/581: General Relativity
Schwarzschild Solution

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(Dated: March 27, 2024)

I. FINDING EXACT SOLUTIONS TO EINSTEIN'S EQUATION.

A general technique to find a solution to the Einstein equation is as follows:

1. Define a coordinate system spanning the region of spacetime you want to describe. Make use of symmetries as much as possible.
2. Within the established coordinate system, write down a trial metric in terms of unknown functions of these coordinates. Again, use the symmetries to eliminate as many metric components as possible.
3. Substitute the trial metric into the Einstein Equation.
4. Solve the resulting differential equations for the remaining unknown functions of the coordinates.

Note that since we have to settle on a coordinate system *and* write down a trial metric *before* solving the Einstein equation, it is extremely difficult to assign physical meaning to the coordinates beforehand.

II. SCHWARZSCHILD SOLUTION

[**A unique vacuum spherically symmetric solution**] The Schwarzschild solution is the *unique* spherically-symmetric solution to the Einstein equation in vacuum. The vacuum part means that it is a solution to

$$R_{\mu\nu} = 0. \tag{1}$$

This means that the solution we are looking for will be valid only *outside* a spherically-symmetric massive body. Another thing we demand is that the metric asymptotes to that of flat spacetime once we are very far away from that massive body (after all, Minkowski spacetime is a solution to Eq. (**??**)).

[**Spherical coordinates and isotropy**] Since we are looking for a spherically-symmetric solution, we might as will start with our usual spherical coordinates (t, r, θ, ϕ). Note however that the physical meaning of some of these coordinates will be different than in flat spacetime. The problem now is to find the 10 independent components of the metric $g_{\mu\nu}(t, r, \theta, \phi)$. Imposing spherical symmetry makes two demands on the structure of the metric. The first, is that the meaning of solid angles should be preserved

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \tag{2}$$

that is, the coefficient of the $d\phi^2$ term should be $\sin^2\theta$ times that of the $d\theta^2$ term. It also means that none of the metric components can depend on θ and ϕ explicitly, as this would break isotropy. Thus we know that we are looking for $g_{\mu\nu} = g_{\mu\nu}(t, r)$.

[**No off-diagonal angular components**] Spherical symmetry also tells us that we cannot have off-diagonal time-angle components such as $g_{\theta t}$ and $g_{\phi t}$ since these are not symmetric under the change $d\phi \to -d\phi$ and $d\theta \to -d\theta$, which break spherical symmetry. For the same reason, we cannot have off diagonal radial-angle terms like $g_{r\theta}$ and $g_{r\phi}$. So, we are left with

$$ds^2 = g_{tt}dt^2 + 2g_{rt}drdt + g_{rr}dr^2 + C^2(r,t)r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where we have introduced the function $C(r, t)$ within $g_{\theta\theta}$ and $g_{\phi\phi}$, which respects the spherical symmetry of the problem. However, we can immediately eliminate this function by redefining the radial coordinate $\bar{r} \equiv Cr$. Let me just drop the "bar" on r for notational simplicity, and simply write

$$ds^2 = g_{tt}dt^2 + 2g_{rt}drdt + g_{rr}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4}$$

[**Diagonalizing the time component**] Similarly, we can redefine our time coordinate $t \to t'(t, r)$ to eliminate the off-diagonal term g_{rt} (see Box 23.1). We are left with

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

[**Solving the Einstein equation**] We are now ready to plug this trial metric into Einstein's equation in vacuum. To do so, we need to compute the Ricci tensor for this metric (see Box 23.2). From the tr term, we have

$$R_{tr} = \frac{1}{B}\frac{\partial B}{\partial t} = 0, \qquad \to \qquad \frac{\partial B}{\partial t} = 0. \tag{6}$$

For the tt and rr components, the following combination is simplest to consider

$$\frac{B}{A}R_{tt} + R_{rr} = \frac{1}{r}\left(\frac{1}{A}\frac{\partial A}{\partial r} + \frac{1}{B}\frac{\partial B}{\partial r}\right) = 0, \qquad \to \qquad \frac{1}{A}\frac{\partial A}{\partial r} = -\frac{1}{B}\frac{\partial B}{\partial r}. \tag{7}$$

The $\theta\theta$ term yields

$$R_{\theta\theta} = -\frac{r}{2AB}\frac{\partial A}{\partial r} + \frac{r}{2B^2}\frac{\partial B}{\partial r} + 1 - \frac{1}{B} = 0. \tag{8}$$

We can take Eq. (**??**) and plug it into Eq. (**??**) to solve for $B(r)$ (see Box 23.3). This yields

$$\frac{1}{B(r)} = 1 + \frac{C}{r}. \tag{9}$$

Since B is independent of time, this means A is of the form

$$A(r,t) = f(t)a(r). \tag{10}$$

Thus

$$\frac{1}{A}\frac{\partial A}{\partial r} = \frac{1}{a}\frac{da}{dr} = -\frac{1}{B}\frac{dB}{dr} \tag{11}$$

Since we know B, we can solve for $a(r)$ (see Box 23.4). This yields,

$$a = K\left(1 + \frac{C}{r}\right) \tag{12}$$

and the metric takes the form:

$$ds^2 = -Kf(t)\left(1 + \frac{C}{r}\right)dt^2 + \frac{dr^2}{1 + C/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{13}$$

[**Redefining the time coordinates**] We can redefine the time coordinate one final time to absorb the factor of $Kf(t)$, that is, $t_{\text{new}} = \sqrt{Kf(t)}t_{\text{old}}$. We then find

$$ds^2 = -\left(1 + \frac{C}{r}\right)dt^2 + \frac{dr^2}{1 + C/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

[**Determining the constant** C] This is the final form of the metric. The constant C can be determined by comparing the time-time component of the above metric with the weak field limit that we discussed last time. There, we saw that

$$g_{tt} = -1 - 2\Phi, \tag{15}$$

where Φ is the Newtonian gravitational potential. Outside a spherically-symmetric body of mass M at a distance r from its center, the gravitational potential is

$$\Phi = -\frac{GM}{r}. \tag{16}$$

We then have

$$-1 - \frac{C}{r} = -1 + \frac{2GM}{r}, \qquad \to \qquad C = -2GM. \tag{17}$$

Thus,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{18}$$