PHYS 480/581: General Relativity
Schwarzschild Properties

Prof. Cyr-Racine
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I. THE SCHWARZSCHILD METRIC

The Schwarzschild solution is the *unique* spherically-symmetric solution to the Einstein equation in vacuum. The vacuum part means that it is a solution to

$$R_{\mu\nu} = 0. \tag{1}$$

This means that the solution we are looking for will be valid only *outside* a spherically-symmetric massive body. The metric takes this form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

[**Radial distance**] As should be pretty clear from the form of the above metric, the radial coordinate r doesn't correspond to the distance between the origin and some spacetime event. Indeed, moving along a radial line ($d\phi = d\theta = dt = 0$) the distance between two events A and B is

$$d_{AB} = \int_{r_A}^{r_B} \frac{dr}{\sqrt{1 - 2GM/r}} > r_B - r_A \tag{3}$$

[**See box 9.1.**] To compute the result exactly, we make the change of variable $u = 2GM/r$, which implies that

$$du = -\frac{2GM}{r^2} dr \qquad \Rightarrow \qquad dr = -\frac{r^2}{2GM} du = -\frac{(2GM)^2}{2GMu^2} du = -\frac{2GM}{u^2} du. \tag{4}$$

We thus have

$$\begin{aligned}
\Delta s &= -2GM \int_{u_A}^{u_B} \frac{du}{u^2\sqrt{1-u}} \\
&= -2GM \left(-\frac{\sqrt{1-u}}{u} - \tanh^{-1}\sqrt{1-u} \right)_{u_A}^{u_B} \\
&= \left(r\sqrt{1 - \frac{2GM}{r}} + 2GM \tanh^{-1}\sqrt{1 - \frac{2GM}{r}} \right)_{r_A}^{r_B}.
\end{aligned} \tag{5}$$

Now, we want to compute the physical distance between two shells with $r = 3GM$ (or $u = 2/3$) and $r = 10GM$ (or $u = 1/5$), so

$$\begin{aligned}
\Delta s &= 2GM \left(\frac{\sqrt{1-u}}{u} + \tanh^{-1}\sqrt{1-u} \right)_{2/3}^{1/5} \\
&\approx 8.78GM > 7GM.
\end{aligned} \tag{6}$$

In particular, for r close to $r_s = 2GM$, that distance can get very large. This is an indication of strong spacetime curvature. Such strong curvature only occurs near ultra-compact massive objects like neutron stars and black holes. How compact you say? For the mass of the Earth ($\sim 6 \times 10^{24}$ kg), $r_s = 8.8$ mm. That is, to generate large curvature, you would need to compact the mass of the whole Earth into a radius of less than 1 cm. This seems crazy, but this is more or less what happens around neutrons stars, where $r_s \sim 2.2$ km and their actual size is only a factor of a few larger.

[**Time dilation near massive object**] As $r \to r_s$, the rate at which a clock ticks is considerably slower. A clock at rest at a given coordinate radius r will measure a time

$$\Delta\tau = \int \sqrt{-ds^2} = \sqrt{1 - \frac{2GM}{r}} \Delta t \tag{7}$$

We see that the time measured by the clock will agree with the coordinate time interval only when $r \to \infty$. *So, the meaning of the coordinate time variable t is the time measured by a clock at $r \to \infty$.* The equation given above implies that someone spending some time near a black hole will age slower than an observer at infinity. For example, if an observer spends some time at $r = 3GM$ near a black hole, her clock will measure

$$\Delta\tau = \sqrt{1 - \frac{2}{3}}\Delta t = \sqrt{\frac{1}{3}}\Delta t \approx 0.58\Delta t \tag{8}$$

For instance, if the observer at infinity measure the black hole explorer to be gone near the black hole for one year, the latter will only have aged ~ 7 months in the mean time.

[**Gravitational redshift**] A final point, light redshift as it tries to escape the Schwarzschild spacetime

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} \tag{9}$$

where r_E is the coordinate where the light is emitted, and r_R is the coordinate where the light is received. For $r_R > r_E$, the light is shifted towards longer wavelength. See **box 9.4** for connection to Equivalence Principle.

II. REISSNER-NORDSTRÖM SOLUTION

The Schwarzschild solution describes the metric around a spherically-symmetric mass concentration. This solution is unique. There is a closely related solution in which the mass concentration can also carries an electric charge. This leads to the Reissner-Nordström solution, another exact solution to Einstein's equation. Since there is a static electric field surrounding the charge, we are no longer looking for a vacuum solution to the Einstein equation, but rather one for which the stress-energy tensor is dominated by that of an electric field. We saw in a past homework assignment that stress-energy tensor for electromagnetism is

$$T_{\mu\nu} = -\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma}. \tag{10}$$

Note that the trace of this is zero

$$\begin{aligned}
T &= g^{\mu\nu}T_{\mu\nu} \\
&= g^{\mu\nu}\left(-\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma}\right) \\
&= -\frac{1}{4}g^{\mu\nu}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} + g^{\mu\nu}g^{\alpha\gamma}F_{\mu\alpha}F_{\nu\gamma} \\
&= -\frac{1}{4}\delta^{\mu}_{\mu}F^{\alpha\beta}F_{\alpha\beta} + F^{\nu\gamma}F_{\nu\gamma} \\
&= -F^{\alpha\beta}F_{\alpha\beta} + F^{\nu\gamma}F_{\nu\gamma} \\
&= 0.
\end{aligned} \tag{11}$$

So, we are looking for a solution to the equation

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{12}$$

since T is zero. Now, which components of $F_{\mu\nu}$ are nonzero? By symmetry (and using our intuition about point charges), we know that the electric field from the central mass density has to be purely radial. Using the same coordinates as in the Schwarzschild case (t, r, θ, ϕ), this means that

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

Now the stress-energy tensor above also involved $F^{\mu\nu}$ (upper indices) which requires the metric to write down. We take the trial metric to be the same form as in the Schwarzschild case

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{14}$$

Then

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = g^{\mu t} g^{\nu r} F_{tr} \tag{15}$$

$$F^{tr} = g^{tt} g^{rr} F_{tr} = -\frac{1}{A}\frac{1}{B}(-E(r)) = \frac{E(r)}{AB} = -F^{rt} \tag{16}$$

We are now ready to compute the right-hand side of the Einstein equation

$$\begin{aligned} T_{rr} &= -\frac{1}{4} g_{rr} F^{\alpha\beta} F_{\alpha\beta} + g^{\alpha\gamma} F_{r\alpha} F_{r\gamma} \\ &= -\frac{1}{4} B(F^{tr} F_{tr} + F^{rt} F_{rt} + g^{tt} F_{rt} F_{rt} \\ &= -\frac{1}{4} B(-\frac{E^2(r)}{AB} - \frac{E^2(r)}{AB}) - \frac{1}{A} E^2(r) \\ &= -\frac{E^2(r)}{2A}. \end{aligned} \tag{17}$$

Similarly, we get (see homework)

$$T_{tt} = \frac{E^2(r)}{2B}, \qquad T_{\theta\theta} = \frac{r^2 E^2(r)}{2AB} \tag{18}$$

Comparing the rr and tt equations above, it is clear that

$$8\pi G(AT_{rr} + BT_{tt}) = AR_{rr} + BR_{tt} = 0, \tag{19}$$

which immediately implies ($A \neq 0$), that

$$\frac{B}{A} R_{tt} + R_{rr} = \frac{1}{r}\left(\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr}\right) = 0 \tag{20}$$

where we used Eq. (23.7) in Moore. Thus

$$\frac{1}{A}\frac{dA}{dr} + \frac{1}{B}\frac{dB}{dr} = \frac{1}{AB}\frac{d}{dr}(AB) = 0, \tag{21}$$

which implies that AB is constant. Now, since we demand that spacetime becomes flat at infinity, we need both A and B to go 1 as $r \to \infty$, which gives

$$AB = 1, \qquad \Rightarrow \qquad B = 1/A. \tag{22}$$

Now the $\theta\theta$ component of the Einstein equation gives (see homework)

$$-r\frac{dA}{dr} + 1 - A = -\frac{d(rA)}{dr} + 1 = 8\pi G\left(\frac{r^2 E^2(r)}{2}\right) \tag{23}$$

We need an extra equation to solve for $E(r)$. It is advantageous to use Maxwell's equation in curved spacetime and in the absence of charged particles, $\nabla_\nu F^{\mu\nu} = 0$. Taking the t component of this equation yields

$$\begin{aligned} \nabla_\nu F^{t\nu} &= \partial_\nu F^{t\nu} + \Gamma^\nu_{\nu\alpha} F^{t\alpha} \\ &= \partial_r F^{tr} + \Gamma^\nu_{\nu r} F^{tr} \\ &= \partial_r E(r) + (\Gamma^t_{tr} + \Gamma^r_{rr} + \Gamma^\theta_{\theta r} + \Gamma^\phi_{\phi r}) E(r) \\ &= \partial_r E(r) + (\frac{1}{2A}\frac{dA}{dr} - \frac{1}{2A}\frac{dA}{dr} + \frac{1}{r} + \frac{1}{r}) E(r) \\ &= \frac{dE}{dr} + \frac{2}{r} E(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 E(r)\right) = 0, \end{aligned} \tag{24}$$

where we used the fact that E is only a function of the coordinate r to convert the partial to a total derivative, and the Christoffels can be found in Moore (or really, in any GR textbook). Maxwell's equation thus implies

$$r^2 E = \text{constant}. \tag{25}$$

To find the constant, we note that as $r \to \infty$ we would like the solution to reduces to that of electromagnetism in flat spacetime, which implies that the constant is $Q/(4\pi)$ (remember that $c = 1$ here, which automatically sets $\epsilon_0 = \mu_0 = 1$, and results in the electric charge being dimensionless in these units). Thus,

$$E(r) = \frac{Q}{4\pi r^2}. \tag{26}$$

We can then substitute this solution into the equation for $A(r)$ above, which yields

$$\frac{d(rA)}{dr} = 1 - 4\pi G r^2 \left(\frac{Q}{4\pi r^2}\right)^2 = 1 - \frac{GQ^2}{4\pi r^2}. \tag{27}$$

Integrating on both sides give

$$\begin{aligned} rA &= r + \frac{GQ^2}{4\pi r} + C \\ A(r) &= 1 + \frac{GQ^2}{4\pi r^2} + \frac{C}{r}, \end{aligned} \tag{28}$$

where C is a constant of integration. Now this solution must reduces to the Schwarzschild solution in the limit that $Q \to 0$, which means that $C = -2GM$. Thus, the Reissner-Nordström solution to the Einstein equation takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{29}$$

This metric has an interesting horizon structure (see homework). This metric implies that black holes cannot have an arbitrary large charge, i.e. $Q^2 \leq 4\pi G M^2$.