PHYS 480/581: General Relativity
Review of Special Relativity

Prof. Cyr-Racine
(Dated: January 22, 2024)

Last time, we discussed the Equivalence Principle, i.e. the impossibility of distinguishing between being in a gravitational field and being in a uniformly accelerating frame, *in a small enough region of spacetime*. The statement of the EEP we saw last time was that the laws of physics reduce to those of Special Relativity (SR) in such a small region. We better review SR to make sure we know what this means.

I. SPECIAL RELATIVITY

A. Spacetime

[**Inseparable 4D continuum and natural units**] In SR and GR, the 3 space coordinates (e.g. $(,x,y,z)$ in cartesian) and the time coordinate t form an *inseparable* 4D continuum (mathematically, a manifold). We cannot consider time and space separately. An individual point in spacetime is called an **event**. Other extended objects like lines, planes, etc. are sets of events. This 4D spacetime in which SR applies is usually referred to as **Minkowski space**.

[**Reference frame**] To compute anything practical in such a spacetime we need to establish a reference frame, which allows us to quantify an event's location using a coordinate system. This is a set of arbitrary labels that we assigned to every point in spacetime. For example, in a cartesian coordinate system, spacetime points are labelled by the 4-tuple (ct,x,y,x), where c is the speed of light. In what follows and for the rest of this course, we will always use natural units for which $c = 1$. Our spacetime points will thus be denoted by (t,x,y,z), assuming cartesian coordinates. Of course, nothing stops us from using other coordinate systems (e.g. spherical coordinates) from describing the events within a given reference frame.

[**Worldline**] A particle existing in spacetime draws a curve through it called a **worldline**, which is a parameterized set of events. Note that such a worldline exists even when the particle is at rest in a given reference frame; in this case the worldline points purely in the time direction.

[**Events and worldlines exist independent of coordinate systems**] It is really important to emphasize that events and worldlines (and more generally vectors and tensors, more on that later) exist independent of any coordinate system (or reference frame). A given event can be described in a multitude of different reference frames, each using their own coordinate system.

B. Inertial Reference Frames

An inertial frame (or inertial reference frame (IRF) as Moore calls them) is a frame of reference in which a body with zero net force acting on it does not accelerate, i.e. such a body is either at rest or moving at constant velocity. Inertial frames are sometime referred to as "free-falling" frames, in these sense that these are frames that follow their "natural" trajectory through spacetime.

[**Example of inertial reference frame**] An approximate example of an inertial (or free-falling) frame is the International Space Station (ISS). As it orbits the Earth, the ISS is following its natural path through spacetime, and what we call "gravity" on Earth is essentially absent within the ISS. If there was no window, the astronauts onboard would have a hard time saying whether they are in empty space away from any massive object or orbiting the Earth (à la Equivalence Principle). Note that the ISS is however not a perfect IRF because of the effects of the Sun and the Earth's atmosphere.

[**Local inertial frame**] In this course, we will be interested in *local* inertial frames, which are defined in a small region of spacetime.

[**Principle of relativity**] The reason why inertial frames are so important is the fundamental axiom of SR (i.e. the principle of relativity), which states that:

The laws of physics are the same in all inertial frames.

We will need to be a little more quantitative about what we mean by the "laws of physics are the same" down the road, but for now you can understand it as saying that all inertial observers (i.e. those located in inertial frames) making the same "well-defined" measurement will agree on the outcome. We will need a little more technology to explain what we mean by "well-defined".

C. Lorentz Transformations

As per the principle of relativity, we need a way to relate different inertial reference frames (IRFs) to each other in orther to verify that the "laws of physics" are indeed the same in all of them. Given an IRF S with coordinate axes $\{t, x, y, z\}$ (using cartesian here), we can define an infinite family of other IRFs, denoted S', though either

- a spacetime translation

$$(t', x', y', z') = (t + \Delta t, x + \Delta x, y + \Delta y, z + \Delta z), \tag{1}$$

  or

- a Lorentz transformation

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \mathbf{\Lambda} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}, \tag{2}$$

  where $\mathbf{\Lambda}$ is a 4×4 matrix. The Lorentz transformations comprise both the standard 3D rotations and the Lorentz boosts (the transformation between two frames that have a constant relative velocity). This last one can be thought of as a spacetime rotation mixing the space and time coordinates (see **box 2.4**). Taken together these form 6 different *linear* transformations (3 rotations and 3 boosts). The matrices $\mathbf{\Lambda}$ form a group under matrix multiplication called the Lorentz group. Once we have a little more technology at hand, we will expand more on the structure of this group as it can tell us what kind of *fundamental objects* can live in such a spacetime.

Note that the combination of translations and Lorentz transformations form the **Poincaré group**, which has 10 elements (4 translations, 3 rotations, and 3 boosts).

[**Example of Lorentz transformations**] Let us give some explicit expressions for Lorentz transformations. For example, a boost along the positive x-axis between frames S' and S takes the form

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}, \tag{3}$$

where $\beta = v/c\,(= v$ in natural units) and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{4}$$

Another example is a rotation around the z-axis by an angle ϕ

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & \sin\phi & 0 \\ 0 & -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}. \tag{5}$$

[**Smooth connection to the identity**] Note that the above transformations reduce to the identity matrix in the limit that $\beta \to 0$ and $\phi \to 0$, respectively. This is in fact a very important property of the Lorentz transformations as it imposes important restrictions on the structure of the matrices $\mathbf{\Lambda}$.

D. Spacetime interval

Since translations and Lorentz transformation are **linear** transformations, coordinate differences between two events labelled 1 and 2, $(\Delta t, \Delta x, \Delta y, \Delta z) = (t_2 - t_1, x_2 - x_1, y_2 - y_1, z_2 - z_1)$ transform the same way under a Lorentz transformation as the coordinates themselves. For example, under a boost along the positive x-axis, the above coordinate difference transforms as

$$\begin{bmatrix} \Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}. \tag{6}$$

[**Simultaneity of events**] Importantly, this means that observers in different IRFs will not agree whether two events are simultaneous or not. For example, if $\Delta t = 0$ in frame S, we still have $\Delta t' = -\gamma\beta\Delta x \neq 0$ in general. **This is where Minkowski spacetime differs significantly from the classical Newtonian picture in which there is an absolute notion of time defined exactly the same everywhere in space.** In SR, observers in different IRFs will in general not agree on the time ordering and spatial separation of events. In this sense, discussing time ordering or spatial separation are not "well-defined" measurements, using the nomenclature introduced above.

[**What observers agree on**] However, all observers will agree on a specific quantity called the **spacetime interval**, which is defined by

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{7}$$

in cartesian coordinates. See **box 2.5** for derivation. The above is an example of a "well-defined" measurement or quantity in SR, and as such, all inertial observers agree on its value. While it is not obvious from the above, Δs^2 has a specific tensorial structure (which ensures that all observers agree on its value) that, say, Δt or Δx alone do not possess. More on that later.

[**Type of intervals**] Depending on the value of Δs^2, we use the following nomenclature to describe the spacetime interval:

- **Spacelike** if $\Delta s^2 > 0$,
- **Lightlike** (or null) of $\Delta s^2 = 0$,
- **Timelike** of $\Delta s^2 < 0$.

Since all observers agree on Δs^2, they will also all agree on the above classification of events.

[**Causality**] While event ordering is frame dependent in SR, there is a strong notion of causality built-in the theory. That is, if event A caused event B, then event A must come before event B in *all* possible IRFs. In particular, event A can only cause event B if $\Delta s^2 \leq 0$ (i.e. their separation is timelike or at most lightlike). In other words, no spacelike-separated events can be causally connected. This will be shown in **box 2.6**.

[**Light cone**] The set of all spacetime points that have a lightlike separation from a particular event A forms its **light cone** (both past and future). The notion of light cone is important as any worldline passing through event A must lie within or on this light cone. The volume within the past light cone of event A contains all the events that could have influenced it, while the volume within the future light cone of A contains all the events that could be influenced by A. See Fig. 1 below.

E. Proper time

As there is not absolute notion of time in SR, but rather an infinite number of inertial reference frames and coordinate systems that can be used to describe events taking place, the coordinate time difference between two events is not what any observers would ever measure on their watch. Instead, the time registered on any observer's watch as they travel in spacetime between two events is called the **proper time** $\Delta\tau$. The proper time is the time elapsed according to a clock attached to a particle traveling in spacetime along its worldline. Consider an observer

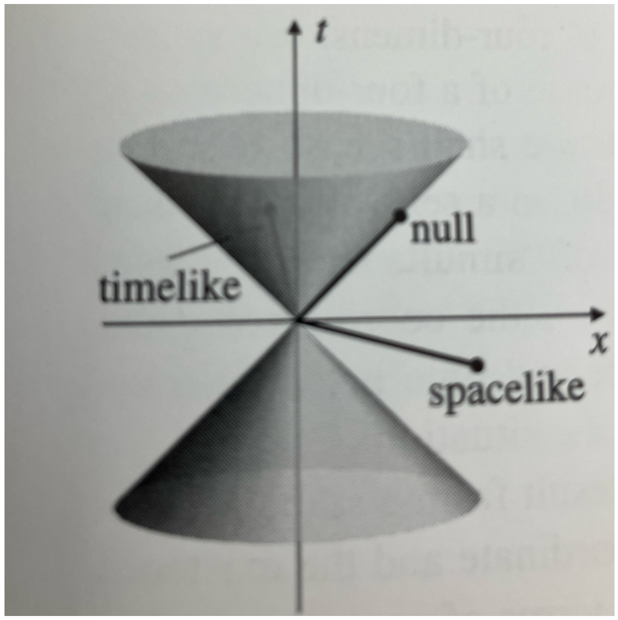

FIG. 1. Illustration of the past and future light cone of an event located at the origin. Reproduced from Carroll (2004).

traveling from point (event) A to point B in spacetime along a specific trajectory (curve) $\mathcal{C}$ (see Fig. 2 below). The proper time that this observer will measure is

$$\Delta\tau_{\mathrm{AB}} = \int_{\mathcal{C}} \sqrt{-ds^2}, \tag{8}$$

where ds^2 is an infinitesimal version of Δs^2, i.e. $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$ (the line element, which we will come back to soon). Note that this integral is path dependent, that is, different observers going from spacetime point A to point B along different trajectories will in general measure different proper time. Also, note that the proper time is in general not equal to the coordinate time, unless the observer is at rest within a given IRF. Interestingly, the *longest proper time* between two events occur for such an observer that is not moving in space.

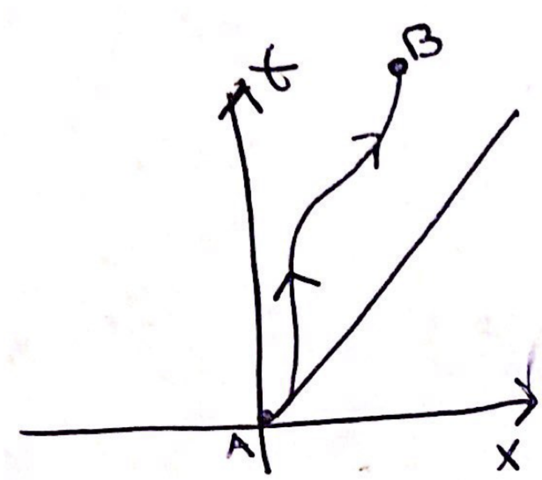

FIG. 2. Observer traveling from event A to event B along a specific spacetime trajectory.