PHYS 480/581: General Relativity
Event Horizon and Global Schwarzschild Structure

(Dated: April 10, 2024)

I. THE SCHWARZSCHILD GEOMETRY AND EVENT HORIZON

Clearly, the Schwarzschild solution has a strange behavior at $r = 2GM$ and $r = 0$.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

- At $r = 2GM$, we have $g_{tt} = 0$ and $g_{rr} \to \infty$. This implies that clocks stop ticking there. Also since light redshifts (its wavelength gets longer) as it moves in the direction of increasing r, light emitted at $r = 2GM$ will get infinitely redshifted according to an observer at $r \gg 2GM$.
- At $r = 0$, we have $g_{tt} \to -\infty$ and $g_{rr} = 0$.

Let's consider these two peculiar points separately.

A. $r = 0$

Consider Lorentz scalars built from the curvature tensor, many possibilities here, e.g. R, $R^{\mu\nu}R_{\mu\nu}$, etc. For example, consider

$$R^{\mu\nu\sigma\rho}R_{\mu\nu\sigma\rho} = \frac{48G^2M^2}{r^6} \tag{2}$$

which is clearly pathological at $r = 0$. Since this is a perfectly fine Lorentz scalar, this result is valid in every possible coordinate system, indicating that this is a *geometric singularity*, that is, a real failure of the spacetime structure at this point. This means that we cannot model an infinitesimal region around $r = 0$ as being locally flat. **In other words, $r = 0$ is not part of the Schwarzschild spacetime manifold.** A theory beyond general relativity is required to describe what is going on at $r = 0$ (we don't yet know this theory, we have candidates for what it could be).

B. $r = 2GM$

There are several hints that what is going there is a *coordinate pathology*, this is, the problem is caused by our choice of coordinates not by a failure of the structure of spacetime. Hints of this can be obtained by asking physical questions that have answers that don't depend on coordinates.

- **Hint 1:** The distance between any radii $r = R > 2GM$ to $r = 2GM$ is finite

$$\Delta s = R\sqrt{1 - \frac{2GM}{R}} + 2GM \tanh^{-1}\sqrt{1 - \frac{2GM}{R}}. \tag{3}$$

- **Hint 2:** The proper time to from from $r = R > 2GM$ to $r = 0$ is also finite (**box 14.2**)

$$\Delta\tau = \frac{\pi R^{3/2}}{\sqrt{8GM}}. \tag{4}$$

- **Hint 3:** Lorentz scalars built from curvature tensor are all well-defined and finite at $r = 2GM$.

So, what happens at $r = 2GM$? First note that $g_{tt} < 0$ and $g_{rr} > 0$ at $r > 2GM$, while $g_{tt} > 0$ and $g_{rr} < 0$ at $r < 2GM$. This means that for $r < 2GM$, r becomes a timelike component, while t becomes spacelike! This is strange to say the least. Now, just like we can't stop the flow of time in flat Minkowski spacetime, we can't stop the flow of the timelike coordinate at $r < 2GM$. **This means that we cannot have $r =$constant trajectories for $r < 2GM$:**

r **has to be changing.** Now this doesn't tell us the *direction* in which the coordinate r is changing, but we suspect that r is shrinking towards $r = 0$. To see this, consider the radial component of the geodesic equation

$$\frac{d^2r}{d\tau^2} = -\frac{GM}{r^2} + \frac{\ell^2}{r^3}\left(1 - \frac{3GM}{r}\right). \tag{5}$$

For $r < 3GM$, we see that the right-hand side is always negative, which implies that $d^2r/d\tau^2 < 0$ there. Note that the above equation has nothing special $r = 2GM$. The facts that r has to keep changing at $r < 2GM$ and that $d^2r/d\tau^2 < 0$ imply that any particle *has to* move *inward*, eventually reaching $r = 0$. So $r = r_s = 2GM$ is a one-way surface called *event horizon*, where information can only flow from $r > r_s$ to $r < r_s$, but never from $r < r_s$ to $r > r_s$.

Interestingly, there is a maximum of proper time that an observer can spend at $r < 2GM$ before hitting the singularity at $r = 0$. This time is πGM (see **box 14.3**).

We can now understand the origin of the coordinate pathology at $r = 2GM$. Remember that we had defined our coordinates to reduce to that of flat spacetime at $r \to \infty$, and that our coordinate t is the time measured by a clock sitting at infinity. However, because of the presence of the event horizon, the observer at infinity cannot access information from $r < r_s$ and thus cannot meaningfully assign time coordinates to these events taking place there (the observer can't synchronize clocks there). For this observer, objects falling towards the event horizon only reach $r = r_s$ at $t \to \infty$. Thus, our coordinate system (t, r, θ, ϕ) cannot describe the entire spacetime structure of the Schwarzschild solution. We need to use other coordinates if we want to describe this whole structure.

II. THE GLOBAL STRUCTURE OF THE SCHWARZSCHILD SPACETIME

Let's introduce a coordinate transformation $(t, r) \to (T, R)$ such that

$$R^2 - T^2 = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \qquad t = 2GM \ln\left|\frac{R+T}{R-T}\right|, \tag{6}$$

and the Schwarzschild metric takes the form

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM}\left(dT^2 - dR^2\right) + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{7}$$

where it is understood that $r = r(T, R)$. These care called the Kruskal coordinates. One of the nice feature of this coordinate system is that T **is always timelike here, while** R **is always spacelike**. Also, note that nothing funny happens in these coordinates at $r = 2GM$.

In these coordinates, a $r =$constant curve corresponds to an hyperbola given by $T^2 - R^2 =$ constant, while a $t =$constant curve is given by

$$T = -CR, \tag{8}$$

where C is a constant. $r = 2GM$ corresponds to $R = \pm T$. We can plot these curves in the $R - T$ plane, as shown in Fig. 1. We can divide this spacetime into 4 distinct regions, as shown in Fig. 2.

- **Region I:** This is the region we've been discussing most so far, the region outside the even horizon of the black hole. This is where our (t, r) coordinates work well to describe the structure of spacetime there.

- **Region II:** This is the region that we think of as the black hole. Once something crosses from region I to region II, there is no return and all trajectories eventually hit the singularity at $r = 0$.

- **Region III:** This is a time-reversed version of region II, a part of spacetime where things can escape to us (in region I), but we can never go there. Everything in this region springs from the past singularity at $r = 0$ and must escape to region I or IV. This is like a "white hole": instead of things having *nothing* escape the event horizon between region I and II (like for a black hole), here *everything* must escape the event horizon between region III and II.

- **Region IV:** Another asymptotically flat region of spacetime, but where time runs in the other direction. Essentially, a mirror image of region I. We cannot reach region IV from region I, nor can observer in region IV reach region I, even in the far future and past.

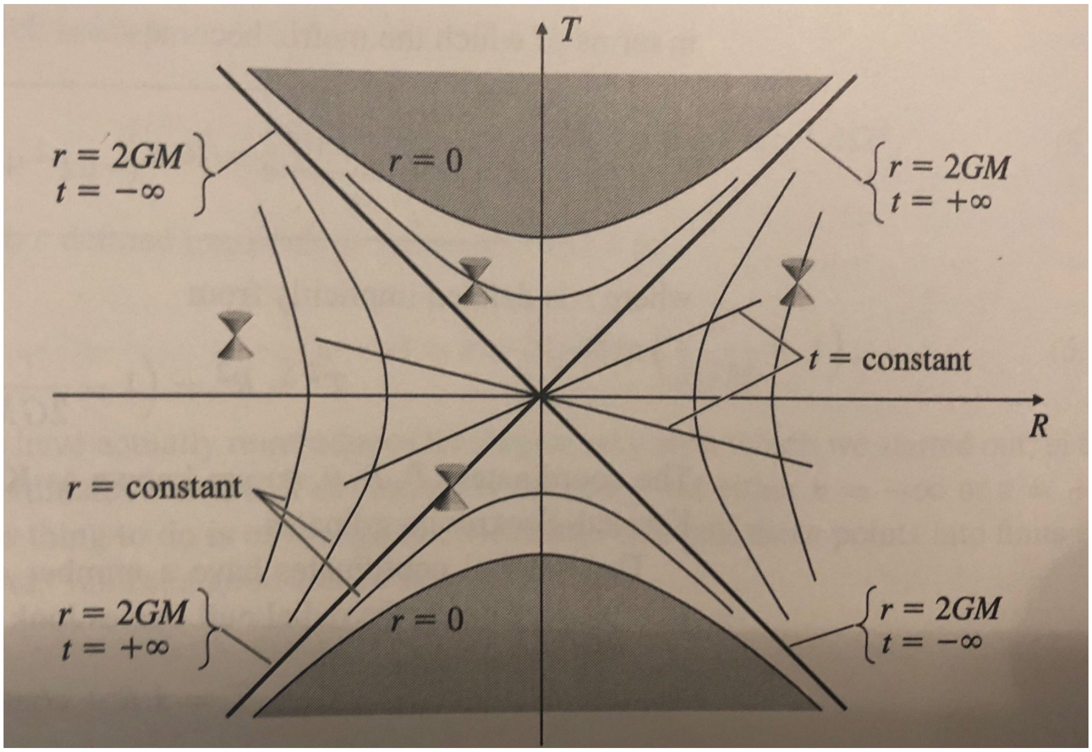

FIG. 1. The Schwarzschild solution in Kruskal coordinates. Figure adapted from Carroll (2003).

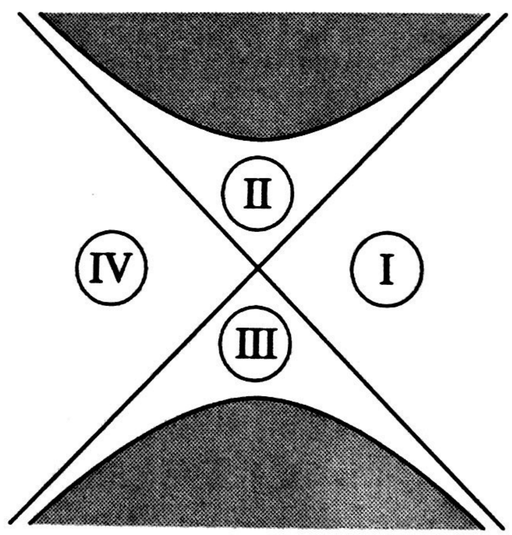

FIG. 2. The different regions of the Schwarzschild spacetime. Figure adapted from Carroll (2003).

It is interesting to consider the global structure of spacelike slices in the Schwarzschild geometry. To do so consider hypersurfaces of constant T as shown in Fig. (3) using the letter A to E. We can draw each T = constant slice, restoring one of the angular direction. These are shown in Fig. 4, with the same A to E labels. For slice A, we clearly have 2 separate regions of asymptotically flat spacetime, each having some curvature as $r \to 2GM$ and having a cone singularity at $r = 0$. For slice B, the two asymptotically flat regions are now connected by a narrow bridge or throat (called an Einstein-Rosen bridge or wormhole). For slice C, the Einstein-Rosen bridge is at its maximum radius of

$r = 2GM$. Slices D and E are similar to slice B and A, respectively, but in the distant future. It's worth emphasizing that the drawings shown in Fig. 4 represent the geometry of the whole spacetime *at an instant in time*. It's not like an observer could move on these surfaces and venture into the wormhole. In fact, you can show that no timelike trajectory can cross an Einstein-Rosen bridge: basically the bridge closes up too fast for any observer to cross from one side to the other.

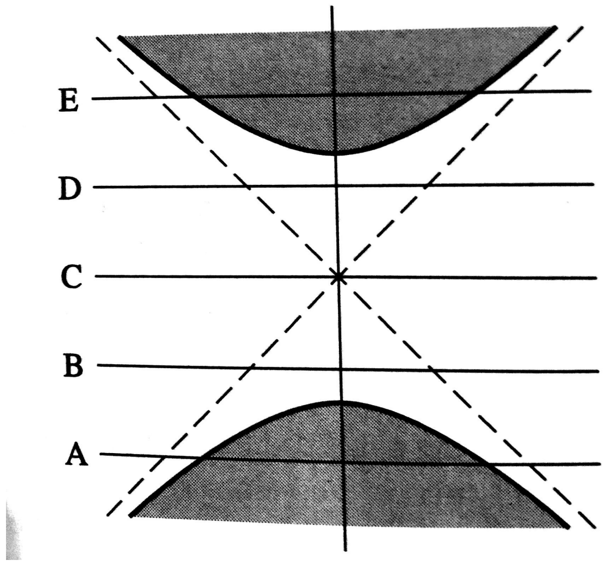

FIG. 3. Spacelike slices in Kruskal coordinates. Figure adapted from Carroll (2003).

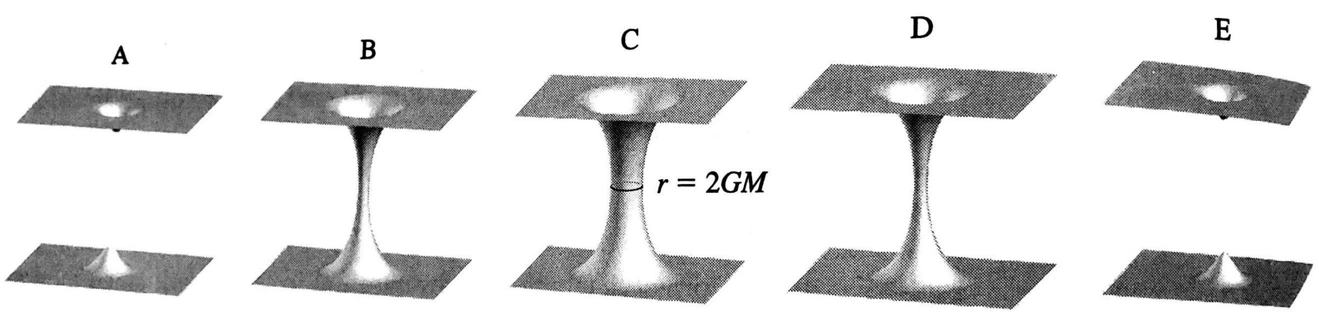

FIG. 4. The geometry of the spacelike surface shown in Fig. 3. Figure adapted from Carroll (2003).