PHYS 480/581: General Relativity
U(1) Gauge Theory as Electromagnetism

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I. $U(1)$ GAUGE THEORY

[**What is $U(1)$?**] You may have heard that electromagnetism is a $U(1)$ gauge theory. Here, $U(1)$ refers to the unitary group of dimension 1, which simply acts on objects by multiplying them by a complex phase $e^{i\theta}$. Because of this, $U(1)$ is often referred to as the "circle" group since θ can be used to denote a position on the unit circle. Electromagnetism is a theory in which important objects are *invariant* under such $U(1)$ phase rotation.

[**A phase factor everywhere in spacetime**] Imagine that at every point in spacetime, we associate a complex phase factor parametrized by $\theta(x)$ (here x denotes spacetime coordinates). This is in complete analogy to associating a tangent vector space to every points in spacetime. As a simple starting point, let set $\theta(x) = 0$ everywhere. Since the phase is the same everywhere, this spacetime is equivalent to standard Minkowski space and I don't have to worry about this phase at every spacetime points. I am free to define functions on this spacetime, such as $\psi_q(x)$, (here, q is a label and not an index, see below) and take their derivatives $\partial_\mu \psi_q$.

[**A spacetime-dependent phase**] Now imagine that I perform a transformation that sets the phase everywhere to be $e^{i\theta(x)}$, where $\theta(x)$ is a smooth non-constant function. The complex phase is now different (though smoothly varying) at every spacetime point. You can think of this transformation as effectively doing a coordinate transformation on the imaginary little circle existing at every point in spacetime. Under this transformation, the function $\psi_q(x)$ picks an extra phase

$$\psi_q(x) \to \psi'_q(x) = e^{iq\theta(x)}\psi_q(x), \tag{1}$$

where q can be thought of as a "response function" for how much the function $\psi_q(x)$ react to a change in $\theta(x)$. As we will see, q turns out to be what we call the electric charge. In particular, functions $\psi_q(x)$ with $q = 0$ are insensitive to the change of phase.

[**Invariance under $U(1)$ transformations**] The theory of electromagnetism is developed by demanding that any physical observables are independent of the choice of phase given in Eq. (1). This is non-trivial, as the presence of the non-vanishing phase factor makes computing derivatives of functions complicated. For instance,

$$\partial_\mu \psi'_q(x) = e^{iq\theta(x)}\partial_\mu \psi_q(x) + iqe^{iq\theta(x)}\psi_q(x)\partial_\mu \theta(x), \tag{2}$$

where the second term is caused by the spatial dependence of the phase $\theta(x)$. If we are demanding that our theory be insensitive to the choice of phase, this term cannot be present when we compute derivatives. The problem with the above is that our differential operator ∂_μ is unaware that nearby points have different phases. We need a differential operator that is aware of this difference.

[**Gauge-covariant derivative**] Consider the following differential operator

$$D_\mu \equiv \partial_\mu + iqA_\mu, \tag{3}$$

where A_μ is a one-form (dual vector) that effectively encodes the difference in $U(1)$ phase between nearby spacetime points. In the above starting point where the phase was vanishing everywhere, we obviously have $A_\mu = 0$. However, once we make the phase transformation given in Eq. (1), we have

$$D_\mu \psi'_q(x) = e^{iq\theta(x)}\partial_\mu \psi_q(x) + iqe^{iq\theta(x)}\psi_q(x)A_\mu + iqe^{iq\theta(x)}\psi_q(x)\partial_\mu \theta(x). \tag{4}$$

If we choose $A_\mu = -\partial_\mu \theta(x)$, we can cancel the two last terms and obtain

$$D_\mu \psi'_q(x) = e^{iq\theta(x)} D_\mu \psi_q(x). \tag{5}$$

Since phase factor can never affect real observables, the above condition is sufficient to claim that the theory is independent of the choice of phase.

[**Gauge connection or potential**] As you can see from the above, the role of the one-form A_μ is to connect values of $\theta(x)$ at nearby points, that is, for $\epsilon^\mu \ll 1$

$$\theta(x+\epsilon) \approx \theta(x) + \epsilon^\mu \partial_\mu \theta\big|_x = \theta(x) - \epsilon^\mu A_\mu(x). \tag{6}$$

Because of this, A_μ is often referred to as a *connection* in the mathematics literature, although physicists usually referred to it as the *gauge potential* or *gauge field*.

[**General gauge transformation**] In the above example, we started with a configuration with $\theta(x) = 0$ everywhere (implying $A_\mu(x) = 0$), and then applied the phase transformation given in Eq. (1). But nothing stops us from starting from a phase configuration with $\theta(x) \neq 0$ (and thus a non-vanishing gauge field $A_\mu(x)$), and make a further phase transformation (usually referred to as a "gauge transformation") with $\theta'(x)$ and related gauge field $A'_\mu(x)$. Under such change, the function $\psi_q(x)$ and $A_\mu(x)$ transform as

$$\psi_q(x) \to \psi'_q(x) = e^{iq\theta'(x)}\psi_q(x), \tag{7}$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \theta'(x). \tag{8}$$

It is easy to see that such transform leaves the gauge covariant derivative of ψ_q invariant (up to a phase)

$$\begin{aligned} D'_\mu \psi'_q(x) &= (\partial_\mu + iqA'_\mu)e^{iq\theta'(x)}\psi_q(x) \\ &= (\partial_\mu + iq(A_\mu - \partial_\mu\theta'(x))e^{iq\theta'(x)}\psi_q(x) \\ &= e^{iq\theta'(x)}\left(\partial_\mu\psi_q + iqA_\mu\psi_q - iq(\partial_\mu\theta')\psi_q + iq\psi_q(\partial_\mu\theta')\right) \\ &= e^{iq\theta'(x)}\left(\partial_\mu\psi_q + iqA_\mu\psi_q\right) \\ &= e^{iq\theta'(x)} D_\mu\psi_q. \end{aligned} \tag{9}$$

[**Gauge field dynamics: Curvature**] The gauge field $A_\mu(x)$ is a dynamical function of spacetime and its equation of motion will thus involve derivatives. The problem though is that terms like $\partial_\mu A_\nu$ are not invariant under the gauge transformation given in Eq. (8). However, the combination

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10}$$

is invariant under that gauge transformation since partial derivatives commute. $F_{\mu\nu}$ is an anti-symmetric $(0, 2)$ tensor (i.e. a two-form). A more formal definition of $F_{\mu\nu}$ can be obtained by considering the commutator of two gauge covariant derivatives and how it acts on a function $\psi_q(x)$.

$$[D_\mu, D_\nu]\psi_q = iqF_{\mu\nu}\psi_q, \tag{11}$$

Now the above commutator says that if I first study the variation of ψ_q in the ν and then in the μ direction, or if I first study the variation of ψ_q in the μ direction and then in the ν direction, I get a nonzero result only if $F_{\mu\nu} \neq 0$. Thus, $F_{\mu\nu}$ tells us about the *curvature* of the internal space described by $\theta(x)$. The label q is interpreted as the electric charge carried by ψ_q.