PHYS 480/581: General Relativity
Intro to GR and the Equivalence Principle

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I. GENERAL RELATIVITY: MAIN CONCEPT

"The physics of gravity is inherently linked to the structure of 4D spacetime."

II. EQUIVALENCE PRINCIPLE: FIRST STEP

Consider the magnitude of the electric force between 2 charges q and Q,

$$F_q = \frac{qQ}{4\pi\epsilon_0 r^2} = m_{\rm I} a, \tag{1}$$

where $m_{\rm I}$ is the mass of the particle carrying charge q, and where we have use Newton's 2nd law in the last equality. Here, $m_{\rm I}$ is the inertial mass (i.e. how strongly it resists being accelerated). Its acceleration is

$$a = \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)\frac{q}{m_{\rm I}} = E_Q\frac{q}{m_{\rm I}}, \tag{2}$$

where E_Q is the electric field created by charge Q. The above implies that particles of different charge will have different trajectory (as they are accelerated differently). Now, contrast this to the magnitude of the gravitational force between a masses M and $m_{\rm g}$

$$F_{\rm G} = \frac{GMm_{\rm g}}{r^2} = \left(\frac{GM}{r^2}\right)m_{\rm g} = m_{\rm I} a, \tag{3}$$

where $m_{\rm g}$ is the gravitational mass, which a priori is not necessarily equal to the inertial $m_{\rm I}$. Here, $m_{\rm g}$ is more like q above (i.e. a "gravitational charge"). Solving for the acceleration in the above, we get

$$a = \left(\frac{GM}{r^2}\right)\left(\frac{m_{\rm g}}{m_{\rm I}}\right). \tag{4}$$

Saying that the ratio in the second parenthesis is equal to 1 is an assumption we make all the time. But is it true?? Why would $m_{\rm g} \neq m_{\rm I}$? Consider for instance a heavy nucleus like gold. This nucleus has a smaller inertial mass than an equal number of protons and neutrons because of the nuclear binding energy between the nucleons. We can measure this inertial mass by accelerating an ionized nucleus in a known electric field. Does the gravitational mass "know" about this binding energy?? Or does it just depend on the number of nucleons?

If $m_{\rm g} = m_{\rm I}$, then every object follows the same trajectory since

$$a = \left(\frac{GM}{r^2}\right)\left(\frac{m_{\rm g}}{m_{\rm I}}\right) = \frac{GM}{r^2}, \tag{5}$$

which is independent of the object's mass. This behavior is of course very familiar to us on Earth since all objects fall with the same acceleration $a \simeq 9.8$ m/s^2. The fact that every objects "falls" the same way has been established experimentally to very high precision. First by Galileo and Newton (1 part in 10^3), then by Eötvös in the late 1880s (1 part in 10^7), and we now know that $m_{\rm I} = m_{\rm G}$ to 1 part in 10^{13}. The equality of the gravitational and inertial mass is often referred to as the **Weak Equivalence Principle (WEP)**.

The WEP implies the existence of a preferred class of trajectories through spacetime (those given by $a = GM/R^2$ on Earth for instance) known as "inertial" of "free-falling" trajectories on which "unaccelerated" objects travel (where "unaccelerated" mean subject to gravity only).

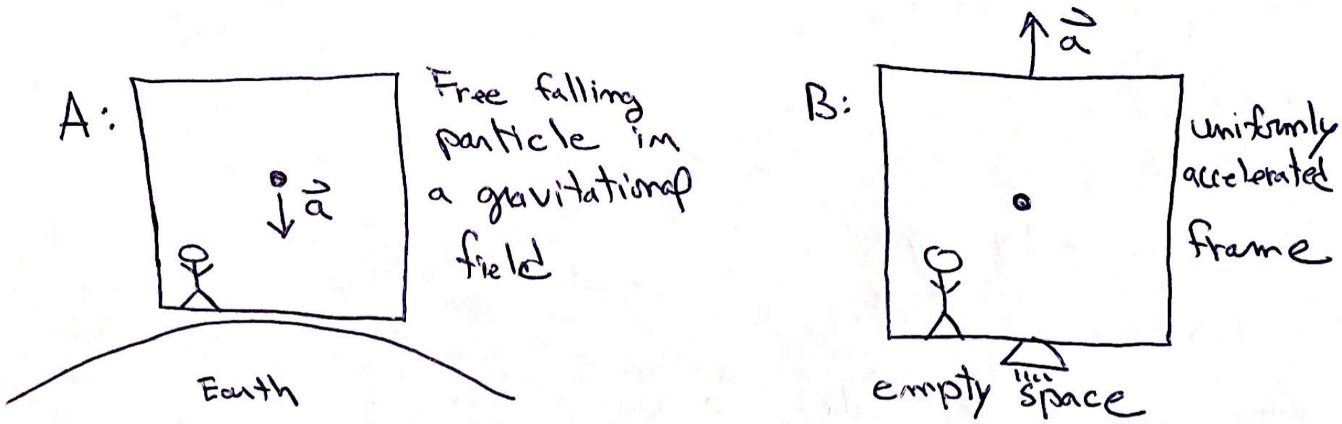

FIG. 1. Key illustration of the Equivalence Principle. In the left panel, an observer is sitting in a closed box on Earth, while in the right panel the observed is a closed box uniformly accelerating.

III. FORMALIZING THE WEAK EQUIVALENCE PRINCIPLE

Consider the similarity of the two situations shown in Fig. 1 above. In both situations, the observer will see the particle accelerate towards the floor of the box. This motivates a more thorough definition of the WEP.

Weak Equivalence Principle: The motion of freely-falling particles are the same in a gravitational field and a uniformly accelerated frame, *in a small enough region of spacetime.*

Note the importance of the last clause about a small enough region of spacetime. In a larger regions of spacetime, there could be inhomogeneities in the gravitational field, which would lead to tidal forces and allowing one to distinguish between the right and left panel in Fig. 1. Also note that the trajectories followed by free-falling objects are called *geodesics* in spacetime.

IV. THE EINSTEIN EQUIVALENCE PRINCIPLE

In Special Relativity (see next lecture), we know that mass is only one side of a more general quantity called energy. For instance, for the gold nucleus considered above, the gravitational field has to couple to the nucleus' binding energy in just the right way to make its gravitational mass come out right. This implies that not only does gravity couple universally to the rest mass of an object, but it must also couple to all forms of energy and momentum universally. We can thus generalize the WEP to include other types of experiments (i.e. not just dropping test masses).

The Einstein Equivalence Principle (EEP): In a small enough region of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of a local experiment (Here, by "laws of physics", we mean "non-gravitational law of physics").

Interestingly, it is possible to write down theories that violate the EEP but not the WEP. For example, a theory in which freely-falling particles begin to rotate as they moved through a gravitational field. Sometimes, we also want to include the "gravitational laws of physics" within our definition of the Equivalence Principle. This leads to the **Strong Equivalence Principle (SEP)**, which is similar to the EEP but also including the "gravitational laws of physics". For instance, a theory in which the gravitational binding energy does not contribute equally to the inertial and gravitational mass of a body would violate the SEP but not the EEP.

So why is the EEP such a big deal? The amazing thing is that it leads to quantitative experimental consequences:

1. Deflection of the apparent positions of background stars when they are near the sun. This was predicted by Einstein in 1907 but the calculation was off by a factor of 2 due to him using Newtonian gravity.

2. The gravitational blueshift or redshift of photons.

Let us focus here on the second consequence involving the shift in wavelength of photons as they travel in a gravitational field. The situation is shown in Fig. 2. According to the EEP, a photon of wavelength λ_0 traveling downward toward the center of the Earth will be indistinguishable from the same photon traveling in a uniformly accelerating spaceship (with the same acceleration as on Earth). Let us analyze the latter situation, assuming non-relativistic velocities.

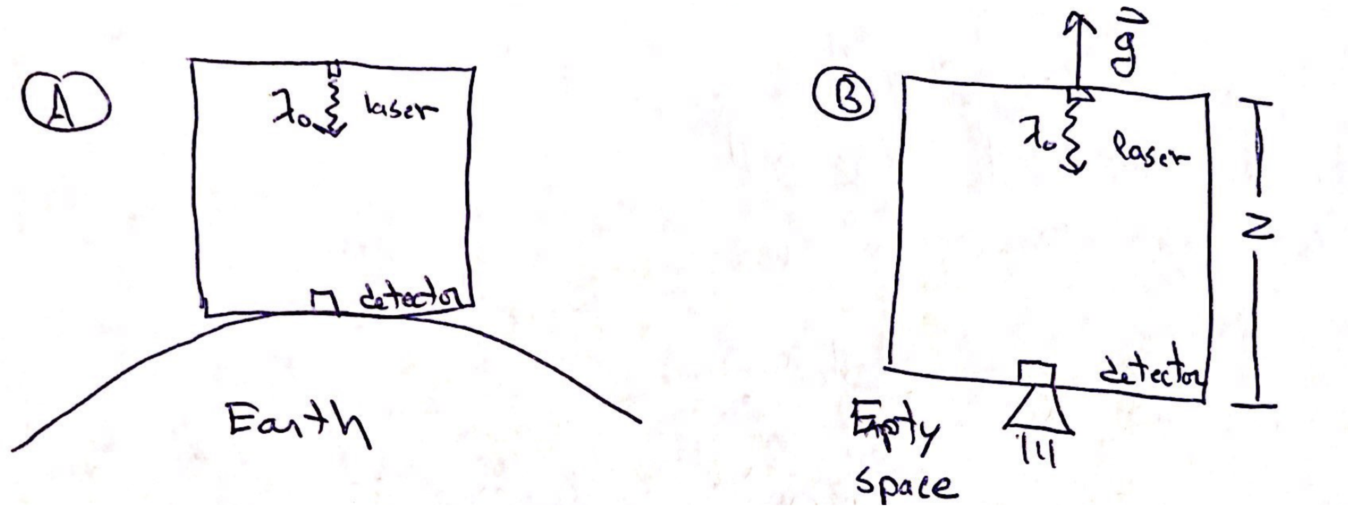

FIG. 2. Comparison between the Pound and Rebka experiment (1959) shown in the left panel, and the identical situation in the right panel according to the EEP.

In the case of the accelerating spacecraft, the time it takes for the photon to travel from the laser to the detector is

$$\Delta t = \frac{z}{c}, \tag{6}$$

where c is the speed of light, and z is the height of the spaceship. During this time, the detector has gained speed

$$\Delta v = g\Delta t = \frac{gz}{c}, \tag{7}$$

where g is the acceleration. Since the detector has a net relative velocity compared to the source at emission, there will be a Doppler shift in the wavelength λ_d of the detected photon

$$\lambda_\mathrm{d} = \sqrt{\frac{1-\Delta v/c}{1+\Delta v/c}}\lambda_0, \tag{8}$$

which implies that $\lambda_\mathrm{d} < \lambda_0$, i.e. the photon is blue-shifted. We can Taylor expand this result for $\Delta v/c \ll 1$ to obtain

$$\frac{\lambda_0 - \lambda_\mathrm{d}}{\lambda_0} \equiv \frac{\Delta\lambda}{\lambda_0} = \frac{\Delta v}{c} = \frac{gz}{c^2}. \tag{9}$$

Now if the EEP holds, we should be able to measure the shift in the photon wavelength as it fall towards the center of the Earth (as in situation A in Fig. 2). This is exactly the experiment that Pound and Rebka did in 1959 at Harvard, confirming the above result gotten using the EEP. Note that the above calculation does not use any the details of GR; it is a pure consequence of the EEP.

V. GRAVITY AND THE EINSTEIN EQUIVALENCE PRINCIPLE

If the EEP is true, is gravity entirely fictitious? Absolutely not!! Gravity is real! It is just not the usual downward force that we call "gravity". Let us abandon for a moment the "small enough region of spacetime" that is so essential to the EEP. As illustrated in Fig. 3 below, take 2 particles of mass m, separated by a a distance d, and release them from the same height above the surface of the Earth. Their paths will initially be parallel, but because they are attracted towards the center of the Earth, their separation will start shrinking and their paths will soon cease to be parallel. This effect is caused by subtle changes in the gravitational field at nearby points (essentially the gravitational acceleration $\vec{g}$ is pointing in slightly different direction for nearby points). We say that the two particles are experiencing *tidal effects* in this case. The magnitude of these tidal effects depends on the mass, radius, and density of the Earth, as well as on the initial separation.

This implies that spacetime does not follow the rule of Euclidean geometry (in which parallel lines stay parallel forever), that is, spacetime is curved (i.e. non-Euclidean). This spacetime curvature is the *key concept* in GR. To quote physicist Sir John Archibald Wheeler,

"Spacetime tells matter how to move; matter tells spacetime how to curve."

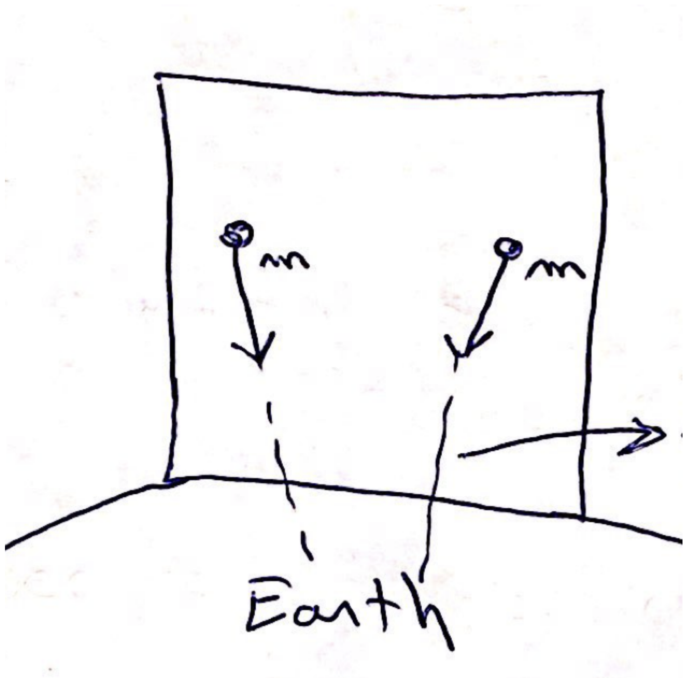

FIG. 3. The reality of gravity: two particles of mass m released from the same height above the surface of the Earth will not follow parallel trajectories. This implies that spacetime is curved.