PHYS 480/581: General Relativity
Curvature Tensor

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I. NOTION OF CURVATURE

[**Parallel transport**] Last time we discussed parallel transporting tensor along specific trajectories in spacetime. As a reminder, a tensor $\boldsymbol{T}$ is said to be parallel transported along the curve $x^\mu(\tau)$ if

$$\frac{dx^\lambda}{d\tau}\nabla_\lambda T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} = 0. \tag{1}$$

[**Parallel transport is path dependent**] Now the key point about parallel transport is that, in general, it is *path dependent*. This path dependence is intrinsically linked to the notion of spacetime *curvature*. For instance, let's say I want to parallel transport a vector $\boldsymbol{V}$ from spacetime point A to spacetime point B. If I choose path 1 parametrized by the curve $x_1^\mu(\tau)$, I get the vector $\boldsymbol{V}_1$. On the other hand, if I choose path 2 parametrized by the curve $x_2^\mu(\tau)$, I get the vector $\boldsymbol{V}_2$. If $\boldsymbol{V}_1 \neq \boldsymbol{V}_2$, then spacetime is said to be curved.

[**Example on a sphere**] A two-dimensional example of this is shown in Fig. 1, where we parallel transport a vector from the equator to the pole of a 2-sphere. On path 1, we simply move the vector directly from the equator to the pole. On path 2, we first transport the vector along the equator, and then move it up to the pole. The resulting vectors at the pole from choosing path 1 and path 2 are clearly different. This matches our intuitive notion that the surface of a sphere is curved.

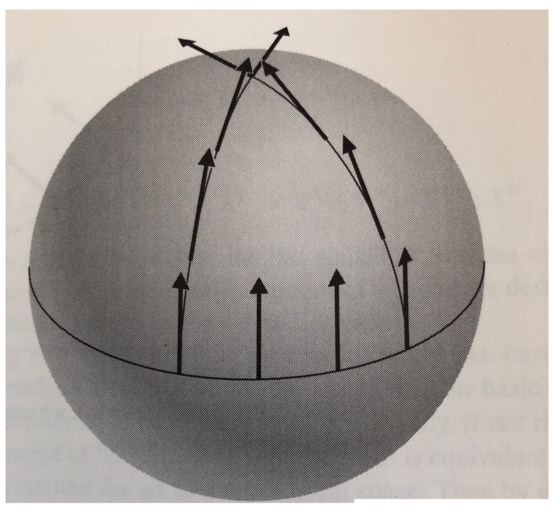

FIG. 1. Parallel transport of a vector from the equator to the pole of a 2-sphere along two different paths. Figure taken from Carroll (2003).

[**Curvature as a commutator**] Moving vectors over different paths as shown in Fig. 1 provides an illustrative example of the curvature of space. However, we would like to have a *local* definition of curvature that doesn't depend on two faraway points A and B. Instead, consider moving the vector $\boldsymbol{V}$ on two infinitesimal paths. For the first path, first move the vector from position x^μ to $x^\mu + \epsilon^\mu$, and then move it to position $x^\mu + \epsilon^\mu + \phi^\mu$, where ϵ^μ and ϕ^μ are infinitesimal constant vectors. For the second path, first move the vector from position x^μ to $x^\mu + \phi^\mu$, and then move it to position $x^\mu + \phi^\mu + \epsilon^\mu$. Let's work through this slowly. For a vector translated from x^μ to $x^\mu + \gamma^\mu$ (where γ^μ can be either ϵ^μ or ϕ^μ), we have

$$V^\sigma(x^\mu + \gamma^\mu) \approx V^\sigma(x^\mu) + \gamma^\nu\nabla_\nu V^\sigma(x^\mu) + \frac{1}{2}\gamma^\nu\nabla_\nu\gamma^\alpha\nabla_\alpha V^\sigma(x^\mu) + \ldots, \tag{2}$$

where we have kept terms up to second order in the small vector γ^μ. This is basically a Taylor expansion in curved space. Applying this expression recursively to $x^\mu \to x^\mu + \epsilon^\mu \to x^\mu + \epsilon^\mu + \phi^\mu$ along path 1, we get

$$V_1^\sigma(x^\mu+\epsilon^\mu+\phi^\mu) = V^\sigma(x^\mu)+\epsilon^\nu\nabla_\nu V^\sigma(x^\mu)+\frac{1}{2}\left(\epsilon^\nu\nabla_\nu\epsilon^\alpha\nabla_\alpha + \phi^\nu\nabla_\nu\phi^\alpha\nabla_\alpha\right)V^\sigma(x^\mu)+\phi^\nu\nabla_\nu\left[V^\sigma(x^\mu) + \epsilon^\mu\nabla_\mu V^\sigma(x^\mu)\right], \quad (3)$$

where we have kept terms up to second order in the small vectors ϵ^μ and ϕ^μ. Along path 2, we move the vector along $x^\mu \to x^\mu + \phi^\mu \to x^\mu + \phi^\mu + \epsilon^\mu$ and we get

$$V_2^\sigma(x^\mu + \phi^\mu + \epsilon^\mu) = V^\sigma(x^\mu) + \phi^\nu\nabla_\nu V^\sigma(x^\mu) + \frac{1}{2}\left(\phi^\nu\nabla_\nu\phi^\alpha\nabla_\alpha + \epsilon^\nu\nabla_\nu\epsilon^\alpha\nabla_\alpha\right)V^\sigma(x^\mu) + \epsilon^\mu\nabla_\mu\left[V^\sigma(x^\mu) + \phi^\nu\nabla_\nu V^\sigma(x^\mu)\right]. \quad (4)$$

The difference between these two operations is

$$V_2^\sigma(x^\mu + \phi^\mu + \epsilon^\mu) - V_1^\sigma(x^\mu + \epsilon^\mu + \phi^\mu) = \epsilon^\mu\phi^\nu\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)V^\sigma = \epsilon^\mu\phi^\nu\left[\nabla_\mu, \nabla_\nu\right]V^\sigma \quad (5)$$

that is, the commutator of two covariant derivatives. Let's expand this commutator

$$\begin{aligned}
\left[\nabla_\mu, \nabla_\nu\right]V^\rho &= \nabla_\mu\nabla_\nu V^\rho - \nabla_\nu\nabla_\mu V^\rho \\
&= \partial_\mu(\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu\nu}\nabla_\lambda V^\rho + \Gamma^\rho_{\mu\sigma}\nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
&= R^\rho{}_{\sigma\mu\nu}V^\sigma,
\end{aligned} \quad (6)$$

where you will fill the intermediate steps in the homework. The equation above can be taken as a definition of the Riemann curvature tensor $R^\rho{}_{\sigma\mu\nu}$, for spacetime without torsion, that is, those for which $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. If any component of this tensor is nonzero, than spacetime is said to be *curved*.

[**Relation to Christoffel connection**] As you will show in the homework, the Riemann tensor can be written in terms of the Christoffel connection as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu}. \quad (7)$$

Since the connection coefficients depend on the first derivative of the metric, then the Riemann tensor depends on the second derivatives of the metric, as well as product of first derivatives of the metric. Note that the product of Christoffel connections in the Riemann tensor means it is nonlinear in the metric. As a consequence, GR is a nonlinear theory, which is one of the reason why it is so difficult to find solutions to Einstein's equation. From its definition in terms of the commutator in Eq. (6), it's clear that Riemann tensor is antisymmetric in its two last indices

$$R^\alpha{}_{\beta\mu\nu} = -R^\alpha{}_{\beta\nu\mu}. \quad (8)$$

We will discuss other symmetries of the Riemann tensor next time.

[**What is flat spacetime?**] If the Riemann tensor vanishes, we know that our spacetime is flat. Since flat spacetime can always be described by the Minkowski metric, which has constant components, we can make the following two statements:

1. If there is a coordinate system in which the components of the metric are constant, than the Riemann tensor vanishes
2. If the Riemann tensor vanishes, we can always construct a coordinate system in which the components of the metric are constant.