PHYS 480/581: General Relativity
The Einstein Equation

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I. LINKING SPACETIME CURVATURE TO ITS ENERGY CONTENT

Last time, we saw how to pack the matter/energy content of a spacetime into a tensorial object, the symmetric stress-energy tensor $T^{\mu\nu}$. We are now ready to see how this matter/energy in turns causes the spacetime to *curve*. From our discussion of the Riemann and Ricci tensors, we know that the curvature of a given spacetime with metric $g_{\mu\nu}$ depends on the second-derivative of the metric. So, very schematically, we are looking for an equation of the form

$$[\nabla^2 g]_{\mu\nu} \propto T_{\mu\nu}. \tag{1}$$

This is of course not a valid tensor equation, but rather a suggestion for what we are looking for. Further guidance is provided by demanding that the above schematic equation reduces to the Poisson equation for the Newtonian gravitational potential in the nonrelativistic limit

$$\nabla^2\Phi = 4\pi G\rho, \tag{2}$$

where G is Newton's gravitational constant and ρ is the mass density here. So, we are looking for a proper tensorial equation linking the second derivative of the metric to the stress-energy tensor. Let's write this equation as

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{3}$$

where $G_{\mu\nu}$ is a symmetric rank 2 tensor depending the the second derivative of the metric and κ is a constant to be determined. Fortunately, we already know a rank 2 tensor depending on the second derivative of the metric: the Ricci tensor $R_{\mu\nu}$. However, since $\nabla^\mu T_{\mu\nu} = 0$, we must also have $\nabla^\mu R_{\mu\nu} = 0$, which is unfortunately not true. You can show that in fact,

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R, \tag{4}$$

which is nonvanishing in general. Here, R is the Ricci scalar. However, the above equation shows that if we were to define

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{5}$$

then we immediately have $\nabla^\mu G_{\mu\nu} = 0$. The tensor $G_{\mu\nu}$ defined above, referred to as the *Einstein tensor*, is the *only* rank 2 symmetric tensor you can build out of the Riemann tensor that is linear in the second derivatives of the metric, does not contain higher derivatives of the metric, is zero in flat spacetime, and is invariant under coordinate transformations. Up to the constant κ, the Einstein equation takes the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{6}$$

The constant κ can be determined by taking the nonrelativistic limit of the Einstein equation and making sure it reduces to Eq. (2) above. This gives $\kappa = 8\pi G$. We thus get

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{7}$$

This equation is sometime written in a slightly different form [Box 21.3]. Contracting both sides with the inverse metric $g^{\mu\nu}$, we get

$$R - \frac{1}{2}R\delta^\mu_\mu = -R = 8\pi G T, \tag{8}$$

where $T = g^{\mu\nu}T_{\mu\nu}$ is the trace of the stress-energy tensor. Substituting this in Eq. (7), we get

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}). \tag{9}$$

In vacuum where $T_{\mu\nu} = 0$, this reduces to $R_{\mu\nu} = 0$, a rather simple form of Einstein equation.

II. THE COSMOLOGICAL CONSTANT

Since $\nabla^\mu g_{\mu\nu} = 0$ always, we are always free to add a term proportional to the metric to either side of the Einstein equation. If we add it to the left-hand side, this is usually referred to as the *cosmological constant* term

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{10}$$

Here, Λ is a constant. Historically, Einstein added this term (with a negative sign) to ensure the Universe as a whole was static. Of course, nowadays we know the Universe is not static and is in fact expanding at an accelerating rate. This acceleration means that Λ (or something like it) is required to be there, albeit with the opposite sign of what Einstein had in mind. In this case, we usually add the Λ term on the right-hand side of the equation such that it contributes to $T_{\mu\nu}$ with

$$-\rho_{\mathrm{vac}} g_{\mu\nu}, \tag{11}$$

where

$$\rho_{\mathrm{vac}} = \frac{\Lambda}{8\pi G}, \tag{12}$$

is usually referred to as the *vacuum energy.* For our purpose, the term "cosmological constant" and "vacuum energy" can be used interchangeably.

III. WEAK-FIELD, NONRELATIVISTIC NEWTONIAN LIMIT

Above, we have taken the constant $\kappa = 8\pi G$. Let's see how this arises by taking the Newtonian limit of Eq. (9). The Newtonian limit is defined by three requirements: the particles are moving slowly compared to the speed of light, the gravitational field is weak (small perturbation around flat space), and the field is constant in time. The second point allows us to write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{13}$$

where $h_{\mu\nu}$ is small, in the sense that we can ignore terms that are quadratic (or higher order) in $h_{\mu\nu}$. The inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{14}$$

where

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} h_{\rho\sigma}. \tag{15}$$

Since only particles moving close to the speed of light have significant pressure, then the stress-energy tensor in this case is

$$T_{\mu\nu} = \rho u_\mu u_\nu. \tag{16}$$

Here, ρ could be the mass density of a star or Earth. In the rest frame of this object, the four-velocity is simply $u^\mu = (1, 0, 0, 0)$, which implies that

$$u_0 = g_{0\mu} u^\mu = g_{00} u^0 = (\eta_{00} + h_{00}) u^0 \simeq \eta_{00} = -1, \tag{17}$$

where we have used the fact that u_0 will multiply ρ which is already small (since spacetime is close to flat). Thus

$$T_{00} = \rho \tag{18}$$

is the dominant component of the stress-energy tensor. The trace of the stress-energy tensor is then

$$T = g^{\mu\nu} T_{\mu\nu} = \eta^{00} T_{00} = -T_{00} = -\rho. \tag{19}$$

The 00 component of Eq. (9) is then

$$R_{00} = \kappa(T_{00} - \frac{1}{2} T g_{00}) = \kappa(\rho - \frac{1}{2}(-\rho)(-1)) = \frac{1}{2}\kappa\rho. \tag{20}$$

Now, what is R_{00}? We have

$$R_{00} = R^{\lambda}{}_{0\lambda 0} = R^{i}{}_{0i0}. \tag{21}$$

Using the definition of the Riemann tensor we have

$$R^{i}{}_{0j0} = \partial_j \Gamma^{i}_{00} - \partial_0 \Gamma^{i}_{j0} + \Gamma^{i}_{j\lambda}\Gamma^{\lambda}_{00} - \Gamma^{i}_{0\lambda}\Gamma^{\lambda}_{j0}. \tag{22}$$

Since we have static field, the second term vanishes. The third and fourth terms are of the form $(\Gamma)^2$ and since the Christoffels are first order in the derivative of $h_{\mu\nu}$ then these terms will be second order. Only the first term survives.

$$\begin{aligned} R_{00} &= \partial_i \Gamma^{i}_{00} \\ &= \partial_i \left[\frac{1}{2} g^{i\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00})\right] \\ &= -\frac{1}{2}\delta^{ij}\partial_i \partial_j h_{00} \\ &= -\frac{1}{2}\nabla^2 h_{00}. \end{aligned} \tag{23}$$

So, we get

$$\nabla^2 h_{00} = -\kappa\rho. \tag{24}$$

The final step is to relate h_{00} to the Newtonian gravitational potential Φ. From Newton's second law ($\vec{F} = m\vec{a}$), the motion of a particle in a gravitational field is

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi, \tag{25}$$

or in components

$$\frac{d^2 x^i}{dt^2} = -\partial_i \Phi. \tag{26}$$

But the equation of motion of the particle should also be given by the gedoesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^{\mu}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0, \tag{27}$$

where we used the fact that the particle moves slowly, which implies

$$\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}, \tag{28}$$

to neglect terms proportional to $dx^i/d\tau$. From above, we already know that

$$\Gamma^{\mu}_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}. \tag{29}$$

Taking the i component of the geodesic equation,

$$\frac{d^2 x^i}{d\tau^2} - \frac{1}{2}\partial_i h_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{30}$$

Now, dividing both sides by $(dt/d\tau)^2$, which converts the τ derivatives to t derivatives, we obtain

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial_i h_{00}. \tag{31}$$

Comparing to the above, we see that $h_{00} = -2\Phi$. Equation (24) above thus takes the form

$$\nabla^2 \Phi = \frac{1}{2}\kappa\rho, \tag{32}$$

which if we compare to Eq. (2) above, implies that $\kappa = 8\pi G$.