PHYS 480/581:General Relativity
Properties of the Einstein Equation

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I. COORDINATE INVARIANCE AND ENERGY-MOMENTUM CONSERVATION

The Einstein equation

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \qquad \text{or} \qquad R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \tag{1}$$

represents 10 equations in 4 spacetime dimensions since both sides are symmetric tensors. Now, the metric $g_{\mu\nu}$ is also a symmetric tensor in four dimensions, which should thus have 10 independent components. So, the Einstein equation should be able to entirely determine the metric tensor.

[**Actual number of degree of freedom**] However, we know that, given a metric describing some spacetime, we can perform a coordinate transformation $x^\mu \to x'^\mu$ and get a perfectly fine metric describing the same spacetime. This means that the Einstein equation can only determine *six* constraints on the metric $g_{\mu\nu}$. How can the 10 components of the Einstein equation only provide 6 constraints on $g_{\mu\nu}$? Because the Einstein tensor satisfies the constraint equation $\nabla_\mu G^{\mu\nu} = 0$, which represents 4 equations, and $10 - 4 = 6$ net equations.

[**Symmetry and energy-momentum conservation**] Last time we demanded that $\nabla_\mu G^{\mu\nu} = 0$ in order to have $\nabla_\mu T^{\mu\nu} = 0$, but note that we don't need to invoke this fact here. Once you define the Einstein tensor as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}, \tag{2}$$

then $\nabla_\mu G^{\mu\nu} = 0$ is guaranteed by the Bianchi identity

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0 \tag{3}$$

(see box 21.2 in Moore). Thus, we can turn the problem around and argue that coordinate invariance implies that $\nabla_\mu G^{\mu\nu} = 0$, which in turns enforces energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$. This is the point of view adopted by most modern practitioners of GR: the fact that Einstein equation is invariant under a general coordinate transformation (diffeomorphism) implies that energy-momentum is (covariantly) conserved. This can be view as an application of Noether's theorem in which the symmetries of the equation of motion lead to conserved quantities.

II. COMMENTS ON THE EINSTEIN EQUATION

[**Nonlinear nature of the Einstein Equation**] As mentioned above, the Einstein equation represents six independent equations (in four dimensions) for the components of the metric $g_{\mu\nu}$. These equations are *differential* equations involving first and second derivatives of the metric as well as the inverse metric. Since the Riemann tensor contains products of Christoffel connections, these equations involve products of metric components and their derivatives, which makes the equations nonlinear.

[**No principle of superposition**] This means that we cannot use the principle of superposition to form new solutions to Einstein's equations from known ones. In addition, the energy-momentum tensor itself depends on the metric tensor (see the perfect fluid case, for example), implying that the metric appears on both sides of the equation. Thus solving the Einstein in all generality is impossible. Finding solutions requires us to assume some symmetries the simplify the equation.

[**Gravity couples to itself**] It is worth discussing the nonlinear nature of Einstein's equation. In Newtonian gravity, if you bring two massive objects nearby, you can simply add together their respective gravitational potential to find the resulting total potential (and thus, the resulting gravitational acceleration) from these two masses. It doesn't work that way in GR: the gravitational field from an object depends on whether other massive objects are nearby.

Basically, in GR the nonlinear nature of the Einstein equation implies that the gravitational field couples to itself, that is, a gravitational field itself can generate a gravitational field!

[**Relation to Equivalence Principle**] This can be thought as a consequence of the Equivalence Principle: if gravitational "energy" did not itself gravitate, then a gravitationally-bound system would have a different inertial mass than gravitational mass. The nonlinear nature of the Einstein equation ensures that the Equivalence Principle is always respected, and it does represent a departure from Newtonian theory. Since the nonlinear behavior becomes more apparent in stronger gravitational fields, you need to go close to massive objects to begin seeing the difference between Newtonian gravity and GR. This is why the orbit of Mercury is most influenced by GR corrections compared to the other planets.

III. WEAK-FIELD, NONRELATIVISTIC NEWTONIAN LIMIT

A. Einstein Equation in the weak-field limit

In the weak-field limit, the Riemann tensor takes the simple form

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right) \tag{4}$$

(see Moore box 22.3) and the Ricci tensor is

$$R_{\beta\nu} = \frac{1}{2}\left(-\eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{\beta\nu} + \partial_\beta H_\nu + \partial_\nu H_\beta\right), \tag{5}$$

where

$$H_\nu = \eta^{\mu\alpha}\left(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\mu}\right). \tag{6}$$

(see box 22.4). Now, we can use our coordinate freedom to send

$$x^\mu \to x^\mu + \xi^\mu, \tag{7}$$

which sends

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu. \tag{8}$$

(see Moore Chap. 30, which we will cover later). This is very similar to a gauge transformation in electromagnetism. We can always choose ξ^μ to set H_ν to zero (4 transformations to set the 4 components of H_ν to zero). In this case, the Einstein equation reduces to

$$-\frac{1}{2}\eta^{\alpha\mu}\partial_\alpha\partial_\mu h_{\beta\nu} = -\frac{1}{2}\Box^2 h_{\beta\nu} = 8\pi G\left(T_{\beta\nu} - \frac{1}{2}\eta_{\beta\nu}T\right), \tag{9}$$

which is valid for a weak gravitational field. For a static source (independent of time), this reduces to

$$\nabla^2 h_{\beta\nu} = -16\pi G\left(T_{\beta\nu} - \frac{1}{2}\eta_{\beta\nu}T\right), \tag{10}$$

which is just the Poisson equation. The formal solution to this equation is

$$h_{\beta\nu}(\vec{r}) = 2\int d^3r_{\mathrm{s}}\frac{G(2T_{\beta\nu} - \eta_{\beta\nu}T)}{|\vec{r} - \vec{r}_{\mathrm{s}}|}, \tag{11}$$

where the integral runs over where the stress-energy tensor has support (i.e. where the matter/energy is). The source term appearing in this equation is always simple in the non-relativistic limit. Starting from the perfect fluid stress-energy tensor $T_{\mu\nu} = (\rho + p)u_\mu u_\nu + g_{\mu\nu}p$, its trace is always $T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3p$. Assuming that the "fluid" is moving slowly, that is $u^t \sim 1$ and $u^i \ll 1$, the source terms are

$$2T_{tt} - \eta_{tt}T \approx \rho + 3p, \tag{12}$$

$$2T_{ti} - \eta_{ti}T = 2T_{ti} \approx -2(\rho + p)u_i, \tag{13}$$

$$2T_{ii} - \eta_{ii}T \approx \rho - p, \tag{14}$$

$$2T_{ij} - \eta_{ij}T \approx 0, \qquad (i \neq j). \tag{15}$$

B. Geodesic equation in weak-field limit

Let's assume a particle is moving non-relativistically in a weak gravitational field. It's four-velocity is approximately given by $u^\mu \approx (1, v_x, v_y, v_z)$, where $v_i \ll 1$ are the standard components of the three-velocity. The spatial components of the geodesic equation are then

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{tt}(u^t)^2 + \Gamma^i_{jk}u^j u^k + 2\Gamma^i_{tj}u^t u^j = 0 \tag{16}$$

Not the term $u^j u^k$ is subdominant compared to the other terms since it is second order in the small velocity. Using $u^t \sim 1$ and realizing that in this limit a τ derivative is equivalent to a t derivative (i.e. $dt/d\tau = 1$), we have

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{tt} + 2\Gamma^i_{tj}v^j \approx 0 \tag{17}$$

Last time, we saw that

$$\Gamma^i_{tt} = -\frac{1}{2}\delta^{il}\partial_l h_{tt}. \tag{18}$$

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$$\begin{aligned}\Gamma^i_{tj} &= \frac{1}{2}\eta^{il}\left(\partial_t h_{jl} + \partial_j h_{lt} - \partial_l h_{tj}\right)\\ &= \frac{1}{2}\delta^{il}\left(\partial_j h_{lt} - \partial_l h_{tj}\right).\end{aligned} \tag{19}$$

The geodesic equation is then

$$\frac{d^2x^i}{dt^2} \approx \frac{1}{2}\delta^{il}\partial_l h_{tt} + \delta^{il}\left(\partial_l h_{tj} - \partial_j h_{lt}\right)v^j. \tag{20}$$

Last time, we identified that $h_{tt} = -2\Phi$, the Newtonian gravitational potential. The second term on the right-hand side has no equivalent in Newtonian mechanics. It is the gravitomagnetic term since this terms is only present if the particle is moving (like a magnetic force). It is convenient to define

$$F_{lj} \equiv \partial_l h_{tj} - \partial_j h_{lt}, \tag{21}$$

which allows us to write down

$$\frac{d^2x^i}{dt^2} \approx \delta^{il}(-\partial_l\Phi + F_{lj}v^j). \tag{22}$$

C. Example with a moving thin solid rod

Let's take a very long thin solid rod of mass density λ (per unit length) extending along the z-axis (we can neglect the pressure of the rod). The rod is moving in the z direction with a small coordinate speed $V \ll 1$, making it essentially a static source. We would like to compute the equation of motion for a non-relativistic particle propagating near this rod. We thus need to compute $\partial_l h_{tt}$ and $\partial_l h_{tj}$. From the formal solution to the Poisson equation above, we have

$$h_{tt} = 2\int d^3r_{\rm s}\frac{G(2T_{tt} - \eta_{tt}T)}{|\vec{r} - \vec{r}_{\rm s}|} \tag{23}$$

It is easiest to do the integral in cylindrical coordinates. In this case, $|\vec{r} - \vec{r}_{\rm s}| = \sqrt{r^2 + z^2}$, and $2T_{tt} - \eta_{tt}T = \lambda\delta^{(2)}(\vec{r})$. We thus get

$$h_{tt}(r) = 2G\lambda\int_{\infty}^{\infty} dz\frac{1}{\sqrt{r^2 + z^2}}. \tag{24}$$

Similarly,

$$h_{tz}(r) = -4G\lambda V\int_{\infty}^{\infty} dz\frac{1}{\sqrt{r^2 + z^2}} = -2Vh_{tt}(r), \tag{25}$$

with $h_{tx} = h_{ty} = 0$. Now the integrals we have above are technically infinite, but this is ok as what we want is the spatial derivative of these integrals. In particular,

$$\begin{aligned}
\partial_r h_{tt} &= 2G\lambda \int_{\infty}^{\infty} dz \partial_r \left(\frac{1}{\sqrt{r^2+z^2}} \right) \\
&= -2G\lambda \int_{\infty}^{\infty} dz \frac{r}{(r^2+z^2)^{3/2}} \\
&= -2G\lambda r \left(\frac{z}{r^2\sqrt{r^2+z^2}} \right)_{-\infty}^{\infty} \\
&= -2G\lambda r \left(\frac{1}{r^2} - \frac{-1}{r^2} \right) \\
&= -\frac{4G\lambda}{r}.
\end{aligned} \tag{26}$$

Similarly, we get

$$\partial_r h_{tz}(r) = \frac{8G\lambda V}{r} \tag{27}$$

Remembering that $\partial_x r = x/r$, $\partial_y r = y/r$, and $\partial_z r = 0$ in cylindrical, we thus have

$$\begin{aligned}
\frac{d^2x}{dt^2} &= \frac{1}{2}\partial_x h_{tt} + \left(\partial_x h_{tj} - \partial_j h_{xt}\right) v^j \\
&= \frac{1}{2}\partial_r h_{tt}\partial_x r + v^z \partial_r h_{tz}\partial_x r \\
&= -\frac{2G\lambda x}{r^2} + \frac{8G\lambda V v^z x}{r^2}.
\end{aligned} \tag{28}$$

Similarly

$$\frac{d^2y}{dt^2} = -\frac{2G\lambda y}{r^2} + \frac{8G\lambda V v^z y}{r^2}. \tag{29}$$

Finally

$$\begin{aligned}
\frac{d^2z}{dt^2} &= \frac{1}{2}\partial_x h_{tt} + \left(\partial_z h_{tj} - \partial_j h_{zt}\right) v^j \\
&= -\partial_x h_{zt} v^x - \partial_y h_{zt} v^y \\
&= -v^x \partial_r h_{zt}\partial_x r - v^y \partial_r h_{zt}\partial_y r \\
&= -\frac{8G\lambda V}{r^2}\left(x v^x + y v^y\right).
\end{aligned} \tag{30}$$

We note that this is very similar to the Lorentz force law for electromagnetism. In fact, we can write the above as

$$\frac{d^2\vec{x}}{dt^2} = \vec{E}_G + \vec{v} \times 4\vec{B}_G, \tag{31}$$

with

$$\vec{E}_G = -\frac{2G\lambda}{r}\hat{r}, \qquad \vec{B}_G = -\frac{2G\lambda}{r}(\vec{V} \times \hat{r}). \tag{32}$$

Up to the factor of 4 for the "magnetic" term, compare this from the electric and magnetic field of a moving line charge,

$$\vec{E} = \frac{2\lambda}{4\pi\epsilon_0 r}\hat{r}, \qquad \vec{B} = \frac{2\lambda}{4\pi\mu_0 r}(\vec{V} \times \hat{r}). \tag{33}$$

This is why the contribution from Eq. (21) above is often referred to as the gravitomagnetic term.