PHYS 480/581: General Relativity
Linearized Gravity and Gauge Freedom

(Dated: April 15, 2024)

I. PERTURBATION AROUND NEARLY FLAT SPACETIME

Here, we would like understand gravity in the weak regime where spacetime is nearly flat. We've explored this a little bit before when taking the Newtonian limit in order to link the 00 entry of the metric to the Newtonian gravitational potential from classical mechanics. Here, we would like to go beyond that and consider the full tensorial structure of the metric in the weak-field limit. Let's consider a spacetime that is nearly flat up to a small perturbation $h_{\mu\nu}$. The metric can then be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. In this scenario, the inverse metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \tag{2}$$

where

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}. \tag{3}$$

The above expressions are derived assuming that $h_{\mu\nu}$ is a small perturbation, and we can thus keep only terms that are first order in $h_{\mu\nu}$. With this metric definition, we can compute the Riemann tensor, which to first order in the metric perturbation takes the form

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu h_{\alpha\nu} + \partial_\alpha\partial_\nu h_{\beta\mu} - \partial_\alpha\partial_\mu h_{\beta\nu} - \partial_\beta\partial_\nu h_{\alpha\mu}\right). \tag{4}$$

Since the Riemann tensor is already first order in $h_{\mu\nu}$, the Ricci tensor and scalar are simply given by

$$R^{\gamma\sigma} = \eta^{\gamma\beta}\eta^{\sigma\nu}\eta^{\alpha\mu}R_{\alpha\beta\mu\nu}, \tag{5}$$

$$R = \eta^{\alpha\mu}\eta^{\beta\nu}R_{\alpha\beta\mu\nu}. \tag{6}$$

Then, the Einstein equation takes the form

$$\begin{aligned} G^{\gamma\sigma} &= 8\pi G T^{\gamma\sigma} \\ R^{\gamma\sigma} - \frac{1}{2}\eta^{\gamma\sigma}R &= 8\pi G T^{\gamma\sigma} \\ \frac{1}{2}\left(\partial^\gamma\partial_\mu h^{\mu\sigma} + \partial^\sigma\partial_\mu h^{\mu\gamma} - \partial^\gamma\partial^\sigma h - \partial^\mu\partial_\mu h^{\gamma\sigma} - \eta^{\gamma\sigma}\partial_\beta\partial_\mu h^{\mu\beta} + \eta^{\gamma\sigma}\partial^\mu\partial_\mu h\right) &= 8\pi G T^{\gamma\sigma}, \end{aligned} \tag{7}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Moore introduces the "trace-reversed" metric perturbations

$$H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{8}$$

This is called "traced-reversed" because

$$\begin{aligned} H &= \eta^{\mu\nu}H_{\mu\nu} \\ &= \eta^{\mu\nu}\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) \\ &= h - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h \\ &= h - \frac{1}{2}4h \\ &= -h. \end{aligned} \tag{9}$$

In terms of $H_{\mu\nu}$, the Einstein equation simplifies a bit (see **Box 30.3**)

$$\Box^2 H^{\gamma\sigma} - \partial^\gamma\partial_\mu H^{\mu\sigma} - \partial^\sigma\partial_\mu H^{\mu\gamma} + \eta^{\gamma\sigma}\partial_\beta\partial_\mu H^{\mu\beta} = -16\pi G T^{\gamma\sigma}, \tag{10}$$

where $\Box^2 \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ (note that what Moore denotes as $\Box^2$ is simply denoted as $\Box$ in a lot of other references).

II. GAUGE TRANSFORMATION

The metric perturbation $h_{\mu\nu}$ (or $H_{\mu\nu}$) is a symmetric rank-2 tensor, and thus has 10 independent entries. However, we area always free to make a coordinate transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$ to set 4 of these entries to zero. Here, $\xi^{\mu} = \xi^{\mu}(t, x, y, z)$ is a function of spacetime coordinates, and we will assume without loss of generality that this coordinate transformation is infinitesimal, $|\xi^{\mu}| \ll 1$. As we have seen many times, the metric transforms as follows under such coordinate transformation

$$g'_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}. \tag{11}$$

For the coordinate transformation given above, the partial derivatives read

$$\begin{aligned} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} &= \frac{\partial}{\partial x'^{\mu}} \left(x'^{\alpha} - \xi^{\alpha} \right) \\ &= \delta^{\alpha}_{\mu} - \frac{\partial \xi^{\alpha}}{\partial x'^{\mu}} \\ &= \delta^{\alpha}_{\mu} - \frac{\partial \xi^{\alpha}}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \\ &\approx \delta^{\alpha}_{\mu} - \partial_{\beta} \xi^{\alpha} \delta^{\beta}_{\mu} \\ &= \delta^{\alpha}_{\mu} - \partial_{\mu} \xi^{\alpha}, \end{aligned} \tag{12}$$

where we have neglected terms that are second order in ξ^{μ} in going from third to the fourth line. Using this transformation, we can show (**Box 30.4**) that the metric perturbation $h_{\mu\nu}$ transforms as follows

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu}, \tag{13}$$

while the trace-reversed perturbation transforms as

$$H'_{\mu\nu} = H_{\mu\nu} - \partial_{\mu} \xi_{\nu} - \partial_{\nu} \xi_{\mu} + \eta_{\mu\nu} \partial_{\alpha} \xi^{\alpha}. \tag{14}$$

It can be shown that the Riemann tensor (and thus the Einstein equation) is invariant under such coordinate transformations. Thus, the underlying physics is not affected by our specific choice of coordinates (at it should!). In terms of terminology, a choice of infinitesimal vector ξ^{μ} is said to specify a *gauge*, in analogy to electromagnetism, where we are always free to define a new vector potential A_{μ} via the gauge transformation $A'_{\mu} = A_{\mu} - \partial_{\mu} \lambda$ without changing the physics.

A useful gauge is the Lorenz gauge, which is defined by $\partial_{\mu} H'^{\mu\nu} = 0$, which simplifies the equation of motion to

$$\Box^2 H^{\mu\nu} = -16\pi G T^{\mu\nu}, \tag{15}$$

which is just the wave equation.

III. IDENTIFYING PHYSICAL DEGREES OF FREEDOM

The fact that we are always free to make a gauge (coordinate) transformation of the metric without changing the underlying physics means that 4 of the 10 independent entries of the metric are unphysical "gauge" degrees of freedom that need to be eliminated before we can ask meaningful physical questions (such as the distance between two spacetime events). Identifying the physical degrees of freedom within the metric and eliminating the "gauge" modes is a complex issues that has caused endless confusion in the literature from the onset of General Relativity in the early twentieth century all the way to the 1980s. Let's work here through a general example of how one can make a choice of gauge to eliminate the the unphysical "gauge" degrees of freedom. To do so, we start by parameterizing our metric perturbation as follows:

$$h_{00} = -2\Phi \tag{16}$$

$$h_{0i} = w_i \tag{17}$$

$$h_{ij} = 2s_{ij} - 2\Psi \delta_{ij} \tag{18}$$

where Ψ encodes the trace of h_{ij}, and s_{ij} is traceless

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij} \tag{19}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right), \tag{20}$$

and latin indices (e.g., i, j, k, l) represent only spatial components. Here, Φ and Ψ are (Lorentz) scalar functions, w_i are the components of a three-vector, and s_{ij} is a symmetric traceless 3-by-3 tensor. Let's count the number of independent degrees of freedom (d.o.f.) here: Φ and Ψ are two scalar functions (2 d.o.f.), w_i is a vector (3 d.o.f.), and s_{ij} (5 d.o.f.), for a total of 10 d.o.f.

For clarity, the line element in this metric takes the form

$$ds^2 = -(1+2\Phi)dt^2 + w_i(dt dx^i + dx^i dt) + [(1-2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j. \tag{21}$$

We know that 4 of those degrees of freedom are unphysical "gauge" modes. In the homework, you will compute how these different entries transform under a coordinate transformation and specify a gauge to eliminate the four unphysical degrees of freedom.

IV. SCALAR, VECTOR, AND TENSOR DECOMPOSITION

[**Number of physical degrees of freedom in metric**] Of course, there are an infinite number of possible gauges that can be chosen. However, there are some general properties of the metric that will always be true once the gauge modes have been eliminated: the metric can *at most* have 2 scalar d.o.f, 2 vector d.o.f, and 2 tensor d.of. Let's consider how this works.

Since any three-vector field can always be decomposed as a sum of a curl-free and divergenceless parts, we have

$$w^i = \partial^i\lambda + \epsilon^{ijk}\partial_j\zeta_k, \tag{22}$$

where λ is a scalar function and the vector ζ^k is divergenceless, $\partial_k\zeta^k = 0$. So, while we might think that w^i represents 3 *vector* degrees of freedom, it is not the case. The function λ is obviously a scalar degree of freedom, while ζ^k (subject to the condition $\partial_k\zeta^k = 0$) represents two vectors degrees of freedom. Similarly, we can decompose the symmetric tensor s_{ij} as follows

$$s^{ij} = s^{ij}_{\perp} + s^{ij}_{\rm S} + s^{ij}_{||}, \tag{23}$$

where $s^{ij}_{\perp}$ is the transverse part obeying $\partial_i s^{ij}_{\perp} = 0$, $s^{ij}_{\rm S}$ is the solenoidal part which obeys $\partial_i\partial_j s^{ij}_{\rm S} = 0$, and $s^{ij}_{||}$ is the longitudinal part obeying $\epsilon^{jkl}\partial_k\partial_i s_{||}{}^{i}{}_{j} = 0$. This means that the longitudinal and solenoidal parts can be written as follows

$$s_{||ij} = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\Theta, \tag{24}$$

$$s_{{\rm S}ij} = \partial_i v_j + \partial_j v_i, \qquad \partial_i v^i = 0. \tag{25}$$

Here, Θ is a scalar function, while v_i is a three-vector. Note that the transverse part $s^{ij}_{\perp}$ cannot be further decomposed. We are now ready to count our degrees of freedom: we have four scalars (Φ, Ψ, λ, Θ) with one degree of freedom each, we have two transverse vectors (ζ^k, v^k) with two degrees of freedom each, and have one traceless-transverse tensor ($s^{ij}_{\perp}$) which contains two degrees of freedom. We therefore have

$$4\,\text{scalars} + 4\,\text{vectors} + 2\,\text{tensors} = 10\,\text{degrees of freedom}. \tag{26}$$

[**Scalar and vector nature of gauge transformation**] Now, let's examine the gauge transformation $x'^{\mu} = x^{\mu} + \xi^{\mu}$. We can write the transformation vector as

$$\xi^{\mu} = (\xi^0, \xi^i), \tag{27}$$

where ξ^0 is a scalar function, and ξ^i can be decomposed as

$$\xi_i = \partial_i f + \epsilon_{ijk}\partial^j V^k, \tag{28}$$

where f is a scalar function and V^k is a divergenceless vector, $\partial_k V^k = 0$. Thus, the transformation vector ξ^μ contains two scalar d.o.f. (ξ^0, f) and two vector d.o.f (V^k, subject to $\partial_k V^k = 0$). This means that such coordinate transformations can be used to eliminate two of the scalar degrees of freedom, and two of the vector degrees of freedom. This indeed means that the physical metric will indeed have 2 scalar, 2 vector, and 2 tensor degrees of freedom.