PHYS 480/581: General Relativity
Geodesics

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I. THE GEODESIC EQUATION

[**Generalizing Newton's 2nd law**] So far, we have discussed spacetimes for which a metric $g_{\mu\nu}$ allows us to compute small spacetime interval ds^2

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1}$$

Now, we would like to study how test particles move in such spacetimes, that is, if I set a particle at some point p with three-velocity $\vec{v}$, what will its trajectory be? In a sense, we are looking for a generalization of Newton's second law $\vec{F} = m\vec{a}$, where $\vec{F}$ here is purely the gravitational force, that is

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\Phi(t, \vec{x}), \tag{2}$$

where Φ is the Newtonian gravitational potential. Specifically, we are looking for an equation valid in all reference frames, that is, a tensor equation.

[**The action for a single particle**] As we usually do in physics to find the equation of motion for some process, we first write down the action. For a freely-falling particle of mass m, the action is simply the particle's proper time $S = -m\tau = -m\int\sqrt{-ds^2}$. If you are unimpressed by this, note that this expression reduces to what you are used to in classical mechanics in the nonrelativistic limit in flat spacetime.

$$\begin{aligned}
S &= -m\int\sqrt{-ds^2} \qquad (3)\\
&= -m\int\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}\\
&= -m\int\sqrt{dt^2 - d\vec{x}^2}\\
&= -m\int\sqrt{1-\left(\frac{d\vec{x}}{dt}\right)^2}dt\\
&\approx -m\int(1-\frac{1}{2}v^2)dt \qquad (v \ll 1)\\
&= \int(-m+\frac{1}{2}mv^2)dt. \qquad (4)
\end{aligned}$$

Thus, up to a constant (which happens to be the rest mass of the particle) which can't affect the equation of motion, we indeed retrieve the action for a free nonrelativistic particle.

[**Deriving the equation of motion**] Now, we would like to derive the equation of motion for a particle for a particle moving through an arbitrary spacetime described by a generic metric $g_{\mu\nu}$. To do so, we would like to extremize the action above such that $\delta S = 0$, in a variational calculus sense. This will give us the equation of motion for this particle. The solution to this equation will be a worldline $x^\mu(\lambda)$ describing the path of the particle through spacetime. Here, λ is a parameter whose value is increasing as we move along the curve. For notational simplicity, let me introduce the function f as

$$f \equiv g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}. \tag{5}$$

Let me also omit the $-m$ factor in front of the action above since it is just a multiplicative constant that can't affect the equation of motion. With these choices, the action of my particle in an arbitrary spacetime is

$$S = \int\sqrt{-f}d\lambda. \tag{6}$$

Then,

$$\delta S = -\frac{1}{2}\int \frac{1}{\sqrt{-f}}\delta f d\lambda. \tag{7}$$

Now, let's make the very convenient choice that our arbitrary parameter λ is the actual proper time τ measured by a clock moving along with the particle. This makes f very simple

$$f = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = g_{\mu\nu}u^\mu u^\nu = -1, \tag{8}$$

since u^μ is the four-velocity of the particle. With this choice, we then have

$$\delta S = -\frac{1}{2}\int \delta f d\tau = 0. \tag{9}$$

So, we have boiled down the problem to computing δf. Using the chain and product rule, we have

$$\delta f = (\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau} \tag{10}$$

Thus,

$$\begin{aligned}
\delta S &= -\frac{1}{2}\int \delta f d\tau \\
&= -\frac{1}{2}\int \left[(\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma + g_{\mu\nu}\frac{d(\delta x^\mu)}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d(\delta x^\nu)}{d\tau}\right] d\tau \\
&= -\frac{1}{2}\int \left[(\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma - \frac{d}{d\tau}(g_{\mu\nu}\frac{dx^\nu}{d\tau})\delta x^\mu - \frac{d}{d\tau}(g_{\mu\nu}\frac{dx^\mu}{d\tau})\delta x^\nu\right] d\tau \\
&= -\frac{1}{2}\int \left[(\partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\delta x^\sigma - \left((\partial_\sigma g_{\mu\nu})\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\mu - \left((\partial_\sigma g_{\mu\nu})\frac{dx^\sigma}{d\tau}\frac{dx^\mu}{d\tau} + g_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\right)\delta x^\nu\right] d\tau \\
&= \int \left[g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]\delta x^\sigma d\tau = 0,
\end{aligned} \tag{11}$$

where in the third line we have integrated by parts, assuming the variation δx^μ to vanish on the boundary of spacetime. Since $\delta S = 0$, we obtain

$$g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{12}$$

Often, we multiply this by the inverse metric $g^{\rho\sigma}$ to obtain

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{13}$$

It is useful to define the object

$$\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}), \tag{14}$$

called the Christoffel connection or symbol. It is not a tensor in general, but note that $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$. With this simplified notation, we obtain

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{15}$$

This is the *geodesic equation* describing the worldline of a particle propagating freely through an arbitrary spacetime with metric $g_{\mu\nu}$. Note that the metric enters exclusively through the Christoffel connection $\Gamma^\rho_{\mu\nu}$.

II. EXAMPLE IN FLAT SPACETIME

For a flat spacetime with Minkowski metric $\eta_{\mu\nu}$, all the Christoffel connection coefficients are zero since the metric contains only constant numbers (+1 and −1), $\Gamma^{\rho}_{\mu\nu} = 0$. So, the geodesic equation is simply

$$\frac{d^2 x^\rho}{d\tau^2} = 0, \tag{16}$$

which has for general solutions

$$x^\rho = a^\rho \tau + b^\rho, \tag{17}$$

a straight line through spacetime. For instance, consider a particle at $y = z = 0$ moving along the x-axis with speed v_x such that $t = 0$ when $x = 0$. This immediately implies $b^\sigma = 0$ for all σ, and $a^y = a^z = 0$. We are left with

$$t = a^0 \tau, \qquad x = a^x \tau. \tag{18}$$

But since the particle is moving at constant velocity, we also know

$$x = v_x t = \frac{a^x}{a^0} t. \tag{19}$$

Thus with $v_x = a^x / a^0$, the solution to the geodesic equation in flat spacetime matches our intuition.