PHYS 480/581: General Relativity
Gravitational Wave Energy

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I. GRAVITATIONAL WAVE SOLUTION IN VACUUM

Last time, we saw that in vacuum the only propagating degrees of freedom were gravitational waves. These were easiest to describe in the transverse-traceless gauge

$$h_{\mu\nu}^{\rm TT} = A_{\mu\nu} e^{ik_\sigma x^\sigma}, \tag{1}$$

where $A_{\mu\nu}$ is a constant, symmetric $(0,2)$ tensor obeying

$$A_{0\nu} = 0 \tag{2}$$

$$\eta^{\mu\nu} A_{\mu\nu} = 0, \tag{3}$$

together with the condition $k_\sigma k^\sigma = 0$ and $k^\mu A_{\mu\nu} = 0$. We also saw that an object get stretched and compressed by a passing gravitational wave according to

$$\frac{\Delta R}{R} \sim A_{+/\times}, \tag{4}$$

where R is the length of the object, and $A_{+/\times}$ is the dimensionless amplitude of the passing gravitational waves, which is typically $A_{+/\times} \sim 10^{-22} - 10^{-18}$.

This week we would like to answer two questions:

1. **How are gravitational wave generated?** To answer this, we will need to move away from the pure vacuum solution and consider how the dynamic of the stress-energy tensor can "shake" spacetime.

2. **What is the energy carried by gravitational waves?** Just light an object emitting light is losing energy, objects emitting gravitational waves will lose energy.

We shall first tackle question 2 here, and consider question 1 next time.

II. GRAVITATIONAL ENERGY IN GENERAL RELATIVITY

The gravitational field carries energy From our experience in electromagnetism, we know that electric and magnetic fields carry energy. In fact, we've computed in a past homework assignment the stress-energy tensor in the presence of electric and magnetic field, and use this tensor to compute the metric around a charged black hole. The gravitational field is no different: it contains stored energy. It's something we are quite familiar from Newtonian gravity where we often compute the gravitational potential energy between two gravitationally-bound objects.

The difficulty of describing gravitational energy in GR In General Relativity, we immediately run into problems trying to discuss the energy stored in the gravitational field. Looking at the Einstein equation

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{5}$$

the stress-energy tensor $T_{\mu\nu}$ appearing on the right-hand side contains all the *non-gravitational* energy and momentum that the spacetime contains. **This tensor tells us nothing about the energy stored in the gravitational field itself (i.e. in the curvature of spacetime)**. That information is instead encoded in the structure of the Einstein tensor $G_{\mu\nu}$, which makes the discussion of the energy "stored" in the gravitational field rather difficult for two main reasons:

- **Non-locality:** General Relativity is a *local* theory, meaning that all objects entering the theory (Einstein tensor, stress-energy tensor, etc.) are all evaluated at one spacetime point. However, discussing the energy stored in the gravitational field require us to consider the curvature of spacetime over an *extended* region of spacetime, which is necessarily nonlocal.

To see this, remember that in an arbitrary curved spacetime, we can always make a local coordinate (gauge) transformation to go to a locally inertial frame in which $\partial_\mu T^{\mu\nu} = 0$, implying that non-gravitational energy and momentum are conserved locally. But if we want the energy stored in the gravitational field to be transferred to a matter component (or vice-versa) then we must consider a larger region of spacetime where the metric is no longer Minkowski throughout, hence abandoning locality.

This also means that **the energy stored in the gravitational field cannot be represented by a local tensor.**

- **Higher-order perturbation theory needed:** In our analysis so far, we have only kept terms in the Einstein equation that were first order in the small metric perturbation $h_{\mu\nu}$. However, the energy stored in the gravitational field is encoded in the **nonlinear terms in the Einstein equation**, i.e. terms that are quadratic and higher-order in $h_{\mu\nu}$. One way to see this is to remember that the energy stored in the gravitational field will itself source spacetime curvature, and that this "sourcing" is necessarily nonlinear. Note that the fact that the energy stored in the gravitational field is at least quadratic in $h_{\mu\nu}$ is consistent with electrodynamics where the energy stored in the field is quadratic in the field (i.e. $T_{00} \propto E^2 + B^2$).

Einstein equation to second-order Now, to first order in perturbation, we saw that the trace-reversed perturbation $H_{\mu\nu}$ satisfies

$$\partial_\alpha \partial^\alpha H_{\mu\nu} = -16\pi G\, T_{\mu\nu} \tag{6}$$

in the Lorenz gauge. At second order in perturbation theory, there will be a significant number of terms that are quadratic in $H_{\mu\nu}$; let's write down their contribution to the Einstein tensor as $2G^{(2)}_{\mu\nu}$. Keeping these contributions to the Einstein equation we have

$$\partial_\alpha \partial^\alpha H_{\mu\nu} - 2G^{(2)}_{\mu\nu} = -16\pi G\, T_{\mu\nu}. \tag{7}$$

Now, we adopt the point of view that $G^{(2)}_{\mu\nu}$ represents the stress-energy contained in the gravitational wave, and move it to the right-hand side of the equation and define

$$T^{\rm GW}_{\mu\nu} \equiv -\frac{\langle G^{(2)}_{\mu\nu} \rangle}{8\pi G}, \tag{8}$$

where the bracket means that we are averaging over a sizable region of spacetime. This is the *effective* stress-energy tensor for the energy stored in gravitational waves. This means we now have

$$\partial_\alpha \partial^\alpha H_{\mu\nu} = -16\pi G\, (T_{\mu\nu} + T^{\rm GW}_{\mu\nu}), \tag{9}$$

as our equation for gravitational wave propagation.

Energy conservation In the Lorenz gauge, we have $\partial_\mu (T^{\mu\nu} + T^{\mu\nu}_{\rm GW}) = 0$, implying that the sum of matter-energy and gravitational energy is conserved.

III. STRESS-ENERGY FOR GRAVITATIONAL WAVES

Let's work out the form of $T^{\rm GW}_{\mu\nu}$ for a + polarized gravitational wave traveling in the z direction. The metric takes the form

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + h_+(t,z) & 0 & 0 \\ 0 & 0 & 1 - h_+(t,z) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{10}$$

where

$$h_+(t,z) = A_+ \cos{(\omega t - \omega z)}. \tag{11}$$

In Box 32.1, the Ricci tensor is computed leading to

$$R^{(2)}_{tt} = R^{(2)}_{zz} = h_+\ddot{h}_+ + \frac{1}{2}\dot{h}_+\dot{h}_+, \tag{12}$$

together with $R^{(2)}_{xx} = R^{(2)}_{yy} = 0$. We now need to discuss spacetime averaging $\langle \ldots \rangle$. The key element is that the derivative of a quantity averaged of a region of spacetime will vanish

$$\langle \partial_\mu X \rangle = 0, \tag{13}$$

that is, we consider a given region of spacetime to be static on average. This implies that

$$\begin{aligned}
\langle R^{(2)}_{tt} \rangle &= \langle h_+\ddot{h}_+ + \frac{1}{2}\dot{h}_+\dot{h}_+ \rangle \\
&= \langle h_+\ddot{h}_+ \rangle + \frac{1}{2}\langle \dot{h}_+\dot{h}_+ \rangle \\
&= \langle \partial_t(h_+\dot{h}_+) - \dot{h}_+\dot{h}_+ \rangle + \frac{1}{2}\langle \dot{h}_+\dot{h}_+ \rangle \\
&= \langle \partial_t(h_+\dot{h}_+) \rangle - \langle \dot{h}_+\dot{h}_+ \rangle + \frac{1}{2}\langle \dot{h}_+\dot{h}_+ \rangle \\
&= -\frac{1}{2}\langle \dot{h}_+\dot{h}_+ \rangle.
\end{aligned} \tag{14}$$

In **Box 32.2**, it is shown that the Ricci scalar at second order in the perturbation is zero

$$\langle R^{(2)} \rangle = \langle \eta^{tt} R^{(2)}_{tt} + \eta^{zz} R^{(2)}_{zz} \rangle = 0. \tag{15}$$

This immediately means that

$$T^{\rm GW}_{tt} = -\frac{\langle G^{(2)}_{tt} \rangle}{8\pi G} = -\frac{\langle R^{(2)}_{tt} \rangle}{8\pi G} = \frac{\langle \dot{h}_+\dot{h}_+ \rangle}{16\pi G}. \tag{16}$$

Now, there is nothing special about the $+$ polarization, and we could also have used the $\times$ polarization. In fact, a $+$ polarization could be transformed into a $\times$ polarization by a 45 degree rotation of our coordinate system. So, for a general gravitational wave which contains both polarization, we must have

$$T^{\rm GW}_{tt} = \frac{\langle \dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times \rangle}{16\pi G}. \tag{17}$$

In **Box 32.3**, it is shown that this can be rewritten in the transverse-traceless gauge

$$T^{\rm GW}_{tt} = \frac{\langle \dot{h}^{\rm TT}_{jk}\dot{h}^{jk}_{\rm TT} \rangle}{32\pi G}. \tag{18}$$

To get this, note that $h_+ = h^{\rm TT}_{xx} = -h^{\rm TT}_{yy}$, and $h_\times = h^{\rm TT}_{xy} = h^{\rm TT}_{yx}$. In all generality, we have

$$T^{\rm GW}_{\mu\nu} = \frac{\left\langle (\partial_\mu h^{\rm TT}_{\rho\sigma})(\partial_\nu h^{\rho\sigma}_{\rm TT}) \right\rangle}{32\pi G}. \tag{19}$$

An interesting result is that the gravitational wave flux (energy transported per unit time per unit area in the direction of the wave's motion), is actually equal to $T^{\rm GW}_{tt}$. This is because $T^{\rm GW}_{tz}$, which is the energy flux, is actually equal to $T^{\rm GW}_{tt}$,

$$T^{\rm GW}_{tz} = T^{\rm GW}_{tt}. \tag{20}$$