PHYS 480/581: General Relativity
Gravitational Wave Production

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I. GRAVITATIONAL WAVE SOLUTION IN THE PRESENCE OF MATTER

To briefly summarize what we have done so far, we have first expanded the metric around flat spacetime,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and where we assume that the deviation from flatness is small. Introducing the trace-reversed metric perturbations

$$H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$, the Einstein equation can be written as

$$\Box^2 H^{\gamma\sigma} - \partial^\gamma\partial_\mu H^{\mu\sigma} - \partial^\sigma\partial_\mu H^{\mu\gamma} + \eta^{\gamma\sigma}\partial_\beta\partial_\mu H^{\mu\beta} = -16\pi G\, T^{\gamma\sigma}. \tag{3}$$

We are free to make a coordinate (gauge) transformation $x'^\mu = x^\mu + \xi^\mu$, under which the trace-reversed metric perturbation transforms as

$$H'_{\mu\nu} = H_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\partial_\alpha\xi^\alpha. \tag{4}$$

We then choose ξ^μ such that $\partial_\mu H'^{\mu\nu} = 0$. This means that ξ^μ satisfies

$$\begin{aligned}
\partial_\mu H'^{\mu\nu} &= \partial_\mu H^{\mu\nu} - \partial_\mu\partial^\mu\xi^\nu - \partial_\mu\partial^\nu\xi^\mu + \eta^{\mu\nu}\partial_\mu\partial_\alpha\xi^\alpha \\
&= \partial_\mu H^{\mu\nu} - \Box^2\xi^\nu - \partial^\nu\partial_\mu\xi^\mu + \partial^\nu\partial_\alpha\xi^\alpha \\
&= \partial_\mu H^{\mu\nu} - \Box^2\xi^\nu \\
&= 0,
\end{aligned} \tag{5}$$

that is, we need

$$\Box^2\xi^\nu = \partial_\mu H^{\mu\nu}. \tag{6}$$

With this choice, which sets the Lorenz gauge, the Einstein equation reduces to

$$\Box^2 H^{\mu\nu} = -16\pi G\, T^{\mu\nu}, \tag{7}$$

where we have omitted the "prime" on the trace-reversed metric perturbation and stress-energy tensor, but it is understood now that this equation is evaluated in the Lorenz gauge. This is a wave equation with a source term. The simplest way to find a solution to this equation is to use the Green's function method. The Green's function $\tilde{G}(x^\beta - x'^\beta)$ satisfies the wave equation in the presence of a delta-function source, that is

$$\Box_x^2\tilde{G}(x^\beta - x'^\beta) = \delta^{(4)}(x^\beta - x'^\beta), \tag{8}$$

where $\Box_x^2$ denote the d'Alembertian operator with respect to the x^β coordinates (instead of the x'^β), and $\delta^{(4)}$ is the four-dimensional Dirac delta function. The general solution to Eq. (7) is then

$$H^{\mu\nu}(x^\beta) = -16\pi G\int\tilde{G}(x^\beta - x'^\beta)T^{\mu\nu}(x'^\beta)d^4x'. \tag{9}$$

It's easy to check that this is indeed a solution to Eq. (7)

$$\begin{aligned}
\Box_x^2 H^{\mu\nu}(x^\beta) &= -16\pi G\,\Box_x^2\int\tilde{G}(x^\beta - x'^\beta)T^{\mu\nu}(x'^\beta)d^4x' \\
&= -16\pi G\int\Box_x^2\tilde{G}(x^\beta - x'^\beta)T^{\mu\nu}(x'^\beta)d^4x' \\
&= -16\pi G\int\delta^{(4)}(x^\beta - x'^\beta)T^{\mu\nu}(x'^\beta)d^4x' \\
&= -16\pi G\, T^{\mu\nu}(x^\beta),
\end{aligned} \tag{10}$$

where we used the sifting property of the Dirac delta function. Now, what is the Green's function of the $\Box^2$ operator? This is derived in multiple references, including Griffiths *Electrodynamics* chap. 10. Here, we will be interested in the *retarded* Green function which takes into account the causality of the emission of gravitational wave (that is, if we observe a gravitational wave here, its source must be in our past light cone)

$$\tilde{G}(x^\beta - x'^\beta) = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta\left(|\mathbf{x}-\mathbf{x}'| - (t-t')\right)\Theta(t-t'), \tag{11}$$

where Θ is the Heaviside step function, that $\mathbf{x}$ is the spatial part of the vector x^β. Plugging this back into Eq. (9), we have

$$\begin{aligned} H^{\mu\nu}(t,\mathbf{x}) &= -16\pi G \int \tilde{G}(x^\beta - x'^\beta) T^{\mu\nu}(x'^\beta) d^4x' \\ &= 4G \int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\left(|\mathbf{x}-\mathbf{x}'| - (t-t')\right)\Theta(t-t') T^{\mu\nu}(x'^\beta) d^4x' \\ &= 4G \int \frac{1}{|\mathbf{x}-\mathbf{x}'|}\Theta(|\mathbf{x}-\mathbf{x}'|) T^{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|, \mathbf{x}') d^3x' \\ &= 4G \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} T^{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|, \mathbf{x}') d^3x', \end{aligned} \tag{12}$$

where we have used the delta function to perform the t' integral and realize that $\Theta(|\mathbf{x}-\mathbf{x}'|) = 1$ always. **Here, the $\mathbf{x}'$ integral runs over all space, but the integrand is only nonzero where $T_{\mu\nu}$ is nonvanishing, that is, where the source of gravitational wave is located.** In general, the sources of gravitational wave (e.g. binary black holes) are very compact compared to the distance at which we measure their gravitational wave signal. This means that

$$|\mathbf{x}-\mathbf{x}'| \simeq |\mathbf{x}| = r, \tag{13}$$

and thus

$$H^{\mu\nu}(t,\mathbf{r}) \approx \frac{4G}{r}\int T^{\mu\nu}(t-r,\mathbf{x}')d^3x'. \tag{14}$$

[**Gravitational waves only involve spatial components**] As we have seen before, the gravitational wave part of the metric perturbations only occurs in the spacelike components $H^{ij}(t,\mathbf{x})$, so let's focus on these components. Consider the following identity, derived using integration by parts (see **Box 33.2**)

$$\int T^{ij} d^3x = \int \partial_k (x^i T^{kj}) d^3x - \int x^i (\partial_k T^{kj}) d^3x, \tag{15}$$

where the first term is a surface integral at infinity (via the divergence theorem), which will vanish since we are assuming that the matter-energy responsible for generating gravitational waves occupies a small region of space. Since we are near flat spacetime, we also have

$$\partial_\mu T^{\mu\nu} = \partial_t T^{t\nu} + \partial_i T^{i\nu} = 0. \tag{16}$$

Putting the two last equations together yields

$$\begin{aligned} \int T^{ij} d^3x &= \int x^i (\partial_t T^{tj}) d^3x \\ &= \frac{1}{2}\frac{d}{dt}\left(\int (x^i T^{tj} + x^j T^{ti}) d^3x\right) \\ &= \frac{1}{2}\frac{d^2}{dt^2}\left(\int x^i x^j T^{tt} d^3x\right), \end{aligned} \tag{17}$$

where in the second line we have used the fact that T^{ij} is symmetric, and the last line can be gotten by applying the same "reverse" integration by parts used in Eq. (15).

$$H^{ij}(t,\mathbf{r}) = \frac{2G}{r}\frac{d^2}{dt^2}\int x'^i x'^j T^{tt}(t-r,\mathbf{x}')d^3x'. \tag{18}$$

[**Making it traceless**] The only remaining issue is that we know that H^{ij} should be traceless since its trace is a scalar degree of freedom which doesn't contribute to the gravitational wave signal. Since T^{tt} is the mass-energy density ρ of the source, removing the trace yields

$$H^{ij}(t,\mathbf{r}) = \frac{2G}{r}\frac{d^2}{dt^2}\int (x'^i x'^j - \frac{1}{3}\eta^{ij}r'^2)\rho(t-r,\mathbf{x}')d^3x'. \tag{19}$$

We now define the *reduced quadrupole moment* of the mass density generating the gravitational waves as

$$\mathcal{I}^{ij}(t) = \int (x'^i x'^j - \frac{1}{3}\eta^{ij}r'^2)\rho(t,\mathbf{x}')d^3x'. \tag{20}$$

Thus, the spacelike metric perturbation far away from the source is

$$H^{ij}(t,\mathbf{r}) = \frac{2G}{r}\ddot{\mathcal{I}}^{ij}(t-r). \tag{21}$$

This tells us that in order to generate gravitational waves, a given mass density ρ needs to have a *dynamic* quadrupole moment (that is, one that change with time). This is referred to as the quadrupole formula.

II. EXAMPLE: ORBITING POINT MASSES

As an example, consider two point masses of equal mass M on a circular orbit of radius $R(t)$ with angular frequency Ω in the xy-plane. Note that Ω is related to M and R via the standard Keplerian relations. Let's labeled the two masses a and b. Their instantaneous position is given by

$$\vec{r}_a = (R(t)\cos\Omega t, R(t)\sin\Omega t, 0), \tag{22}$$

$$\vec{r}_b = (-R(t)\cos\Omega t, -R(t)\sin\Omega t, 0). \tag{23}$$

Since these are point masses, their mass density is described by Dirac delta function

$$T^{tt}(t,\mathbf{x}) = \rho(t,\mathbf{x}) = M\delta(z)\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t) + \delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right). \tag{24}$$

The xx reduced quadrupole moment is then (dropping the primes to reduce clutter)

$$\begin{aligned}
\mathcal{I}^{xx}(t) &= \int (x^2 - \frac{1}{3}(x^2+y^2+z^2))\rho(t,\mathbf{x})dxdydz \\
&= M\int (x^2 - \frac{1}{3}(x^2+y^2))\left(\delta(x-R(t)\cos\Omega t)\delta(y-R(t)\sin\Omega t) + \delta(x+R(t)\cos\Omega t)\delta(y+R(t)\sin\Omega t)\right)dxdy \\
&= 2M\left(R^2(t)\cos^2\Omega t - \frac{1}{3}(R^2(t)\cos^2\Omega t + R^2(t)\sin^2\Omega t)\right) \\
&= 2MR^2(t)\left(\cos^2\Omega t - \frac{1}{3}\right) \\
&= 2MR^2(t)\left(\frac{\cos 2\Omega t}{2} + \frac{1}{6}\right) \\
&= \frac{MR^2(t)}{3}\left(3\cos 2\Omega t + 1\right),
\end{aligned} \tag{25}$$

with similar calculations for the other components. In all, we have

$$\mathcal{I}^{ij}(t) = \frac{MR^2(t)}{3}\begin{bmatrix} (1+3\cos 2\Omega t) & 3\sin 2\Omega t & 0 \\ 3\sin 2\Omega t & (1-3\cos 2\Omega t) & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{26}$$

Thus, the gravitational wave signal seen at a distance r away from the source (using Eq. (21)).

$$H^{ij}(t,\mathbf{r}) = \frac{8GMR^2(t_r)\Omega^2}{r}\begin{bmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

where $t_r \equiv t - r$. **Here, we have assumed that the $R(t)$ is changing very slowly (i.e. much slower than the rotation period $T = 2\pi/\Omega$).** Note that the frequency of oscillation of the gravitational wave is always twice the orbital frequency of the two point masses. Also note that for a wave traveling in the z-direction (but only for this!), the above result is transverse and $H^{ij} = H^{ij}_{\rm TT}$.

III. ENERGY LOSS DUE TO GRAVITATIONAL RADIATION: ORBITAL DECAY

Last time, we saw that the energy density of a gravitational waves is

$$T^{tt}_{\rm GW} = \frac{1}{32\pi G}\Big\langle \dot{h}^{ij}_{\rm TT}\dot{h}^{\rm TT}_{ij}\Big\rangle. \tag{28}$$

where $h^{ij}_{\rm TT}$ is the transverse-traceless metric perturbation. As you will show in the homework, this is also the *flux* of gravitational energy coming out of the source. Now, in vacuum far away from the source we already know that $h^{\mu\nu}_{\rm TT} = H^{\mu\nu}_{\rm TT}$. So, we have

$$h^{ij}_{\rm TT}(t,r) = H^{ij}_{\rm TT}(t,r) = \frac{2G}{r}\ddot{\mathcal{I}}^{ij}_{\rm TT}(t-r), \tag{29}$$

where $\mathcal{I}^{ij}_{\rm TT}$ is the transverse-traceless reduced quadrupole moment. Now, the reduced quadrupole moment is already traceless by construction. But how do we make it transverse? Basically, we need to "project out" the non-transverse part. This is accomplished by a projection operator P^j_m such that

$$P^j_m = \delta^j_m - n^j n_m, \qquad \text{(Note that } P^j_m P^n_j = P^n_m). \tag{30}$$

This projection operator project any vector in the plane perpendicular to the unit vector $\vec{n}$ (here n^j are the spatial components of a unit vector $\vec{n}$). To project the rank-2 reduced dipole moment $\mathcal{I}^{ij}$ into a plane perpendicular to a vector $\vec{n}$, we will need two projection operators (one of each index) such that

$$\mathcal{I}^{jk}_{\rm TT} \equiv \left(P^j_m P^k_n - \frac{1}{2}P^{jk}P_{mn}\right)\mathcal{I}^{mn}. \tag{31}$$

So the flux (power per unit area) of of gravitational wave energy coming out of the source at a distance r away is

$$\text{Flux} = \frac{G}{8\pi r^2}\Big\langle \dddot{\mathcal{I}}^{ij}_{\rm TT}\dddot{\mathcal{I}}^{\rm TT}_{ij}\Big\rangle. \tag{32}$$

Now let's use Eq. (31) to express the flux in terms of $\dddot{\mathcal{I}}^{ij}$ (Box 33.5)

$$\begin{aligned}
\dddot{\mathcal{I}}^{jk}_{\rm TT}\dddot{\mathcal{I}}^{\rm TT}_{jk} &= \left(P^j_m P^k_n - \frac{1}{2}P^{jk}P_{mn}\right)\dddot{\mathcal{I}}^{mn}\left(P^a_j P^b_k - \frac{1}{2}P_{jk}P^{ab}\right)\dddot{\mathcal{I}}_{ab} \\
&= \left(P^a_m P^b_n - \frac{1}{2}(P_{mk}P^k_n P^{ab} + P^{ak}P_{mn}P^b_k) + \frac{1}{4}(P^k_k P_{mn}P^{ab})\right)\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab} \\
&= \left(P^a_m P^b_n - P_{mn}P^{ab} + \frac{1}{4}(P^k_k P_{mn}P^{ab})\right)\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab} \\
&= \left(P^a_m P^b_n - \frac{1}{2}P_{mn}P^{ab}\right)\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab}, \qquad \text{since } P^k_k = 2 \\
&= \left((\delta^a_m - n^a n_m)(\delta^b_n - n^b n_n) - \frac{1}{2}(\delta_{mn} - n_m n_n)(\delta^{ab} - n^a n^b)\right)\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab} \\
&= \dddot{\mathcal{I}}^{ab}\dddot{\mathcal{I}}_{ab} - n^b n_n\dddot{\mathcal{I}}^{an}\dddot{\mathcal{I}}_{ab} - n^a n_m\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{an} + n^a n_m n^b n_n\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab} - \frac{1}{2}n_m n_n n^a n^b\dddot{\mathcal{I}}^{mn}\dddot{\mathcal{I}}_{ab} \\
&= \dddot{\mathcal{I}}^{ab}\dddot{\mathcal{I}}_{ab} - 2n^b n^m\dddot{\mathcal{I}}^a_m\dddot{\mathcal{I}}_{ab} + \frac{1}{2}n^a n^b n^m n^n\dddot{\mathcal{I}}_{ab}\dddot{\mathcal{I}}_{mn},
\end{aligned} \tag{33}$$

where we have used the fact that $\mathcal{I}_{ij}$ is traceless, i.e., $\delta^{ij}\mathcal{I}_{ij} = 0$. To compute the *total* power P emitted by the gravitational wave source, we need to integrate the flux over a sphere of radius r surrounding the source

$$\begin{aligned}
P &= \frac{G}{8\pi r^2}\int r^2\sin\theta d\theta d\phi\Big\langle \dddot{\mathcal{I}}^{ij}_{\rm TT}\dddot{\mathcal{I}}^{\rm TT}_{ij}\Big\rangle \\
&= \frac{G}{16\pi}\int \sin\theta d\theta d\phi\Big\langle 2\dddot{\mathcal{I}}^{ab}\dddot{\mathcal{I}}_{ab} - 4n^b n^m\dddot{\mathcal{I}}^a_m\dddot{\mathcal{I}}_{ab} + n^a n^b n^m n^n\dddot{\mathcal{I}}_{ab}\dddot{\mathcal{I}}_{mn}\Big\rangle,
\end{aligned} \tag{34}$$

where $\vec{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Note that $\mathcal{I}^{ab}$ is independent of θ and ϕ, and these factors can thus be taken out of the integrals. We are left with integral of the type (Box 33.6)

$$\int n^i n^k \sin\theta d\theta d\phi = \frac{4\pi}{3}\eta^{ik}, \tag{35}$$

$$\int n^i n^j n^k n^m \sin\theta d\theta d\phi = \frac{4\pi}{15}(\eta^{ij}\eta^{km} + \eta^{jk}\eta^{im} + \eta^{ik}\eta^{jm}). \tag{36}$$

Performing the contraction in the expression for P yields

$$P = \frac{G}{5}\left\langle \dddot{\mathcal{I}}^{ij}\,\dddot{\mathcal{I}}_{ij}\right\rangle. \tag{37}$$

Going back to our example from the previous section for the two orbiting point masses, we have

$$\dddot{\mathcal{I}}^{ij}(t) = MR^2(t)(2\Omega)^3 \begin{bmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{38}$$

where again we have assumed that $R(t)$ changes very slowly. Then,

$$\begin{aligned} \left\langle \dddot{\mathcal{I}}^{ij}\,\dddot{\mathcal{I}}_{ij}\right\rangle &= M^2R^4(t)(2\Omega)^6\left\langle \sin^2 2\Omega t + \sin^2 2\Omega t + \cos^2 2\Omega t + \cos^2 2\Omega t\right\rangle \\ &= 128M^2R^4(t)\Omega^6, \end{aligned} \tag{39}$$

and thus the power emitted by the orbiting point masses is

$$P = \frac{128}{5}GM^2R^4(t)\Omega^6. \tag{40}$$

Note the sixth (!) power of the orbital frequency appearing in this expression. The power emitted will rob the system from some of its orbital kinetic energy, leading to orbital decay and a shrinking $R(t)$.