PHYS 480/581: General Relativity
Gravitational Waves

(Dated: April 17, 2024)

I. THE TRANSVERSE-TRACELESS GAUGE

We again consider in the weak regime where spacetime is nearly flat and the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. As we did last time, we parameterize the metric perturbation as follows:

$$h_{00} = -2\Phi \tag{2}$$

$$h_{0i} = w_i \tag{3}$$

$$h_{ij} = 2s_{ij} - 2\Psi\delta_{ij} \tag{4}$$

where Ψ encodes the trace of h_{ij}, and s_{ij} is traceless

$$\Psi = -\frac{1}{6}\delta^{ij}h_{ij} \tag{5}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij}\right), \tag{6}$$

and latin indices (e.g., i, j, k, l) represent only spatial components. Here, Φ and Ψ are (Lorentz) scalar functions, w_i are the components of a three-vector, and s_{ij} is a symmetric traceless 3-by-3 tensor. Now, suppose that we perform a gauge transformation $x'^\mu = x^\mu - \xi^\mu$, such that

$$\partial_i s^{ij} = 0, \qquad \partial_i w^i = 0. \tag{7}$$

Note that these are four equations, which will determine the four components of ξ^μ (see homework). You will also show in the homework that in vacuum ($T_{\mu\nu} = 0$), the solution to Einstein equation in this gauge are

$$\Phi = 0, \tag{8}$$

$$\Psi = 0, \tag{9}$$

$$w^i = 0. \tag{10}$$

This means that the metric perturbation takes the simple form

$$h^{\rm TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2s_{ij} & \\ 0 & & & \end{bmatrix} \tag{11}$$

where the TT subscript refers to the "transverse-traceless" gauge, since s_{ij} is traceless. **Important: this tells us that in vacuum, only traceless spatial fluctuation of the metric can exist.** While we could now write the remaining Einstein equation in terms of s_{ij}, it is customary to to write gravitational wave solutions in terms of $h^{\rm TT}_{\mu\nu}$. Note that in this gauge, the trace-reversed perturbation $H_{\mu\nu}$ is equal to metric perturbation itself

$$H^{\rm TT}_{\mu\nu} = h^{\rm TT}_{\mu\nu}, \tag{12}$$

since both perturbations are traceless in this case. Now, the ij component of the Einstein equation in vacuum implies that $h^{\rm TT}_{\mu\nu}$ obeys a wave equation

$$\Box^2 h^{\rm TT}_{\mu\nu} = 0 \tag{13}$$

which is just the wave equation, together with the requirements above

$$h^{\rm TT}_{0\nu} = 0 \tag{14}$$

$$\eta^{\mu\nu}h^{\rm TT}_{\mu\nu} = 0 \tag{15}$$

$$\partial_\mu h^{\mu\nu}_{\rm TT} = 0. \tag{16}$$

Note that the last condition is what Moore calls the Lorenz gauge, but it is really the requirement that the metric perturbations are transverse.

II. GRAVITATIONAL WAVES

Since $h^{\rm TT}_{\mu\nu}$ obeys a wave equation, we know that one set of solutions are plane waves

$$h^{\rm TT}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma}, \tag{17}$$

where $A_{\mu\nu}$ is a constant, symmetric $(0,2)$ tensor obeying

$$A_{0\nu} = 0 \tag{18}$$
$$\eta^{\mu\nu} A_{\mu\nu} = 0. \tag{19}$$

Equation (17) represents moving in the $\vec{k}$ direction with speed $v = \omega/k$, where we have taken $k_\sigma = (\omega, \vec{k})$.The wave equation implies that (**Box 31.1**)

$$\begin{aligned} \Box^2 h^{\rm TT}_{\mu\nu} &= 0 \\ \eta^{\alpha\beta}\partial_\alpha\partial_\beta A_{\mu\nu} e^{ik_\sigma x^\sigma} &= 0 \\ -k_\sigma k^\sigma h^{\rm TT}_{\mu\nu} &= 0, \end{aligned} \tag{20}$$

which implies in all generality that $k_\sigma k^\sigma = 0$. This means that the wavenumber describing gravitational waves are *always* lightlike (or null). **By definition, the means that gravitational waves always follow lightlike trajectories through spacetime with $v = \omega/k = 1$ (remember that the speed of light is set to unity here)**. Finally, the transverse condition implies that (Box 31.1)

$$\begin{aligned} \partial_\mu A^{\mu\nu} e^{ik_\sigma x^\sigma} &= 0 \\ ik_\mu A^{\mu\nu} e^{ik_\sigma x^\sigma} &= 0, \end{aligned} \tag{21}$$

which implies that

$$k_\mu A^{\mu\nu} = 0, \tag{22}$$

which means that the vector k_μ is orthogonal to $A^{\mu\nu}$.

Now, let's pick a particular direction for the wave propagation: the z direction. This means that the wave vector takes the form

$$k^\mu = (\omega, 0, 0, \omega) \tag{23}$$

which automatically satisfies $k_\mu k^\mu = 0$. Now, Eq. (22) implies

$$k_\mu A^{\mu\nu} = -\omega A^{0\nu} + \omega A^{3\nu} = 0 \quad \Rightarrow A^{3\nu} = 0, \tag{24}$$

since $A^{0\nu} = 0$. Now, the traceless-ness condition implies that

$$A_{11} + A_{22} = 0, \tag{25}$$

and we also have $A_{12} = A_{21}$ since $A_{\mu\nu}$ is symmetric. Thus,

$$h^{\rm TT}_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} e^{i(\omega z - \omega t)} \tag{26}$$

What is the motion of a test particle initially at rest in this spacetime $u^\mu = (1,0,0,0)$. Well, you can show that the geodesic equation for such a particle is (**Box 31.3**)

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0. \tag{27}$$

This is telling us that the coordinate of a particle doesn't change, which doesn't tell us much. A more illuminating analysis is to consider what happens to a ring of test particles of radius R initially at rest with respect to each other.

Imagine we place this ring at $z = 0$. Then the displacement of each test particle from the origin is $\Delta x = R\cos\theta$, $\Delta y = R\sin\theta$, and $\Delta z = 0$. The squared distance from the origin is

$$\begin{aligned}\Delta s^2 &= g_{\mu\nu}\Delta x^\mu \Delta x^\nu \\ &= (\eta_{\mu\nu} + h^{\mathrm{TT}}_{\mu\nu})\Delta x^\mu \Delta x^\nu \\ &= (\eta_{\mu\nu} + A_{\mu\nu}\cos\omega t)\Delta x^\mu \Delta x^\nu \\ &= (1 + A_{11}\cos\omega t)\Delta x^2 + (1 + A_{22}\cos\omega t)\Delta y^2 + 2A_{12}\cos\omega t\Delta x\Delta y \end{aligned} \tag{28}$$

Assume that only $A_{11} = A_+$ is non-zero, we then get

$$\begin{aligned}\Delta s^2 &= R^2(1 + A_+\cos\omega t)\cos^2\theta + R^2(1 - A_+\cos\omega t)\sin^2\theta \\ &= R^2(1 + A_+\cos\omega t\cos 2\theta)\end{aligned} \tag{29}$$

Then, if $A_+ \ll 1$, then

$$\Delta s \approx R(1 + \frac{1}{2}A_+\cos\omega t\cos 2\theta) \tag{30}$$

See **Box 31.4** for the other polarization. And the amplitude $A_{+/\times}$ are indeed very small, with $A_{+/\times} \sim 10^{-22} - 10^{-18}$. To give you a sense of how small the effect is, consider how the distance between two mirrors change in the LIGO gravitational wave experiment as a gravitational wave passes by. This is given by

$$\Delta S = \frac{1}{2}RA_{+/\times} \tag{31}$$

Here $R = 4$ km, which gives $\Delta S \sim 10^{-19}$m for $A_{+/\times} = 10^{-22}$. This is 10^4 smaller than the size of a proton! The fact that such shift can be measured is quite a feat of engineering.