PHYS 480/581: General Relativity
Particle Orbits

Prof. Cyr-Racine

(Dated: April 3, 2024)

I. PARTICLE ORBITS IN THE SCHWARZSCHILD METRIC

The Schwarzschild metric takes this form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

We would like to study the trajectories of test particles in motion within such a spacetime geometry. To do this, we will of course use the geodesic equation in this form

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{2}$$

Looking at the $\mu = t$ component, we get

$$\frac{d}{d\tau}\left(g_{t\nu}\frac{dx^\nu}{d\tau}\right) = \frac{d}{d\tau}\left(g_{tt}\frac{dt}{d\tau}\right) = -\frac{d}{d\tau}\left(\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau}\right) = 0, \tag{3}$$

which implies that

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \equiv e = \text{constant}. \tag{4}$$

As $r \to \infty$, the left-hand side just become $dt/d\tau = u^0 = p^0/m$, and so e the relativistic energy per unit mass. Thus, the $\mu = t$ component of the geodesic equation implies that e is constant as the test particle moves around its trajectory. Now looking at the $\mu = \phi$ component, we get

$$\frac{d}{d\tau}\left(g_{\phi\nu}\frac{dx^\nu}{d\tau}\right) = \frac{d}{d\tau}\left(g_{\phi\phi}\frac{d\phi}{d\tau}\right) = \frac{d}{d\tau}\left(r^2\sin^2\theta\frac{d\phi}{d\tau}\right) = 0, \tag{5}$$

which implies that

$$r^2\sin^2\theta\frac{d\phi}{d\tau} \equiv \ell = \text{constant}. \tag{6}$$

Here ℓ is the relativistic angular momentum per unit mass, since in the Newtonian $L/m = r^2 d\phi/dt \approx \ell$ for an equatorial orbit ($\sin\theta = 1$). In **box 10.1**, we will show that trajectories in a Schwarzschild geometry are always planar, which means we can set $\theta = \pi/2$ throughout our discussion without loss of generality. So far, we've established 2 constants of motion for trajectories in a Schwarzschild geometry: **the relativistic energy and angular momentum per unit mass**.

To make further progress, let's look at the constraint from the normalization of the four-velocity, $g_{\mu\nu}u^\mu u^\nu = -1$.

Expanding this, we get

$$\begin{aligned}
-1 &= g_{\mu\nu}u^\mu u^\nu \\
&= g_{tt}(u^0)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2 \\
&= -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \\
&= \left(1-\frac{2GM}{r}\right)^{-1}\left(-e^2 + \left(\frac{dr}{d\tau}\right)^2\right) + \frac{\ell^2}{r^2} \\
-\left(1-\frac{2GM}{r}\right) &= -e^2 + \left(\frac{dr}{d\tau}\right)^2 + \left(1-\frac{2GM}{r}\right)\frac{\ell^2}{r^2} \\
e^2 - 1 &= \left(\frac{dr}{d\tau}\right)^2 - \frac{2GM}{r} + \frac{\ell^2}{r^2} - \frac{2GM\ell^2}{r^3} \\
\frac{1}{2}\left(e^2-1\right) &= \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3} = \text{constant}.
\end{aligned} \tag{7}$$

We can think of the $\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2$ term as the "radial kinetic energy per unit mass", and the other terms as some form of effective potential

$$\tilde{V}(r) = -\frac{GM}{r} + \frac{\ell^2}{2r^2} - \frac{GM\ell^2}{r^3}, \tag{8}$$

where the first term looks like the Newtonian gravitational potential energy, the second term is something like the centrifugal barrier, while the third term has no equivalence in Newtonian physics. The presence of this last term indicates that orbits in the Schwarzschild geometry will behave differently at small radii compared to their Newtonian counterparts. In particular, this last term is responsible (in part) for the precession of Mercury's perihelion.

II. CIRCULAR ORBITS

Taking a τ derivative of Eq. (7), we get

$$0 = \frac{d^2r}{d\tau^2}\frac{dr}{d\tau} + \frac{d\tilde{V}}{dr}\frac{dr}{d\tau} \qquad \rightarrow \frac{d^2r}{d\tau^2} = -\frac{d\tilde{V}}{dr}. \tag{9}$$

Thus, circular orbits occur at radii for which $d\tilde{V}/dr = 0$, as this corresponds to having no radial acceleration. Solving for r yields [**box 10.4**]

$$r_c = \frac{6GM}{1 \pm \sqrt{1 - 12(GM/\ell)^2}} \tag{10}$$

Imposing that the square root at the denominator is real implies that circular orbits can only exist if $\ell > \sqrt{12}GM$. If this condition is satisfied, there will be a (stable) circular orbit a slightly larger radius than $6GM$ (that corresponding to the negative sign), an another (unstable) orbit at a radius slightly smaller than $6GM$. An interesting feature of the above equation is that if we set the angular momentum per unit mass to its smallest possible value ($\ell = \sqrt{12}GM$), than these two solutions merge with $r_c = 6GM$. The bottom line her is that $r_c = 6GM$ corresponds to the Innermost Stable Circular Orbit (ISCO). As we will see in **box 10.6**, this orbit is barely stable as it is an inflection point in the effective potential $\tilde{V}(r)$, rather than a true minimum.