PHYS 480/581: General Relativity
Properties of the Riemann and Ricci tensors

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I. PROPERTIES OF THE RIEMANN TENSOR

[**A remarkable commutator**] Last time, we defined the Riemann curvature tensor via the relation

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma, \tag{1}$$

which is valid for spacetime without torsion, that is, those for which $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. Let's think for a moment how remarkable this relation is. On the right-hand side, we have double covariant derivatives applied on the components of the vector $\boldsymbol{V}$ which is equal on the left to a linear relation with the components of $\boldsymbol{V}$, *independent* of the derivatives of V^σ. From this relation, the physical meaning $R^\rho{}_{\sigma\mu\nu}$ emerges: **it describes how the σ component of a vector gets projected into its ρ component when parallel transported around an infinitesimal loop in the plane given by the μ, ν coordinates**.

[**Nonlinear in the metric**] The Riemann tensor can be written in terms of the Christoffel connection as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\sigma}\Gamma^\sigma_{\beta\mu}. \tag{2}$$

Note that the products of Christoffel connections in the Riemann tensor means it is nonlinear in the metric. As a consequence, GR is a nonlinear theory, which is one of the reason why it is so difficult to find solutions to Einstein's equation.

[**Properties of the Riemann tensor**] From its definition in terms of the commutator in Eq. (1), it's clear that Riemann tensor is antisymmetric in its two last indices

$$R^\alpha{}_{\beta\mu\nu} = -R^\alpha{}_{\beta\nu\mu}. \tag{3}$$

Note that this is also true with all indices lowered

$$R_{\alpha\beta\nu\mu} = -R_{\alpha\beta\mu\nu}. \tag{4}$$

To find the other properties of the Riemann tensor, we can specialize to a locally inertial frame (LIF), where the derivative of the metric at some point p is zero and the metric there is Minkowski. This means that the Christoffel connection coefficients are zero at point p, although their derivatives (which involves second derivatives of the metric) are generally nonzero there. At this point, the Riemann tensor takes the form [**Box 19.1**]

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\mu g_{\alpha\nu} + \partial_\alpha\partial_\nu g_{\beta\mu} - \partial_\beta\partial_\nu g_{\alpha\mu} - \partial_\alpha\partial_\mu g_{\beta\nu}\right) \tag{5}$$

From this, we can read off the other properties of the Riemann tensor. It is antisymmetric in it's two first indices

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \tag{6}$$

and it is symmetric under interchange of the first pair of indices with the second

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \tag{7}$$

You can also show that

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \tag{8}$$

The 3 equations above were derived in a LIF, but they are all valid tensor equations, and thus must be valid in all possible frames. The Riemann tensor also satisfies the Bianchi identity,

$$\nabla_\sigma R_{\alpha\beta\mu\nu} + \nabla_\nu R_{\alpha\beta\sigma\mu} + \nabla_\mu R_{\alpha\beta\nu\sigma} = 0, \tag{9}$$

which is essentially a geometric consistency relation [**box 19.4**].

[**Number of independent components**] Given these symmetries, we can show that in an n-dimensional spacetime, the Riemann tensor has

$$\frac{1}{12}n^2(n^2-1) \tag{10}$$

independent components. In four spacetime dimensions, this means that there are 20 independent components. In 2 space dimensions (like the surface of a two-sphere), there is a single independent component of the Riemann tensor.

II. CONTRACTIONS OF THE RIEMANN TENSOR

We can construct the Ricci tensor by contracting the first and third indices

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}. \tag{11}$$

Using the symmetry of the Riemann tensor, it is straightforward to show that the Ricci tensor is symmetric in its two indices

$$R_{\mu\nu} = R_{\nu\mu}. \tag{12}$$

Let's show that

$$\begin{aligned} R_{\mu\nu} &= R^{\alpha}{}_{\mu\alpha\nu} \\ &= g^{\alpha\beta}R_{\beta\mu\alpha\nu} \\ &= g^{\alpha\beta}R_{\alpha\nu\beta\mu} \\ &= g^{\beta\alpha}R_{\alpha\nu\beta\mu} \\ &= R^{\beta}{}_{\nu\beta\mu} \\ &= R_{\nu\mu}. \end{aligned} \tag{13}$$

We can also take the trace of the Ricci tensor to form the Ricci scalar (or curvature scalar) R

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}. \tag{14}$$

R is real Lorentz scalar, meaning that it is the same in every coordinate systems.

III. INDEPENDENT COMPONENTS OF THE RIEMANN TENSOR

Given these two properties of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}, \tag{15}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}, \tag{16}$$

we need $\alpha \neq \beta$ and $\mu \neq \nu$ in order for the Riemann tensor to be nonzero. In n dimensions, the pair $\alpha\beta$ can admit $n(n-1)$ values. However, not all of these pairs are independent since switching the order of $\alpha\beta$ only flips the sign of the Riemann tensor. So there are $n(n-1)/2$ independent pairs of $\alpha\beta$. The same applies for the $\mu\nu$ pair. So far, we thus have

$$\frac{n^2(n-1)^2}{4} \tag{17}$$

independent components of the Riemann. Now we also have to take into account the symmetry property

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}. \tag{18}$$

This essentially halves the number of independent components, except for the $n(n-1)/2$ "diagonal" elements for which $\alpha\beta = \mu\nu$. So subtracting the diagonal elements from the total, dividing by two, and then adding back the diagonal elements, we obtain

$$\frac{1}{2}\left(\frac{n^2(n-1)^2}{4} - \frac{n(n-1)}{2}\right) + \frac{n(n-1)}{2} = \frac{1}{8}\Big(n(n-1)\left(n(n-1)+2\right)\Big). \tag{19}$$

The last tricky part is to take into account this final property

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0. \tag{20}$$

The above can only be non-zero if $\alpha \neq \beta \neq \mu \neq \nu$. This means that the above only provide additional constraints if $n \geq 4$. Given that all indices need to be different, and that we have complete permutation symmetry, the number of constraints is

$$\frac{n(n-1)(n-2)(n-3)}{4!}, \tag{21}$$

which automatically ensures that no new constraints arise if $n < 4$. Putting everything together we get

$$\frac{1}{8}\Big(n(n-1)\left(n(n-1)+2\right)\Big) - \frac{n(n-1)(n-2)(n-3)}{24} = \frac{1}{12}n^2(n^2-1). \tag{22}$$