PHYS 480/581: General Relativity
The Stress-Energy Tensor

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I. THE STRESS-ENERGY (OR ENERGY-MOMENTUM) TENSOR

So far, we have discussed various aspects of spacetime curvature such as how to determine whether a spacetime is curved and what is the trajectory of a particle in curved spacetime. Now, it is time to introduce what is *causing* the curvature of spacetime in the first place: its energy content. As we've done before, we want to define a tensor that contains *all* information about the matter/energy content of spacetime (i.e. electrons, protons, photons, neutrinos, energy contained in electric and magnetic field, other exotic energies). We call this tensor the "stress-energy" tensor (or sometime the "energy-momentum" tensor), $T^{\mu\nu}$.

[**Meaning the stress-energy tensor**] The stress-energy tensor is symmetric $T^{\mu\nu} = T^{\nu\mu}$. In general, $T^{\mu\nu}$ represents the flux of four-momentum p^μ across a surface of constant x^ν. Specifically, T^{00} stands for the rest-frame energy density, while T^{0i} is the momentum density (momentum per unit volume) in the ith direction (this is sometime referred to as the "energy flux" in the ith direction). T^{ii} gives the ith component of the force (per unit area) by a fluid element in the i-direction: this is what we commonly refer to as the ith component of the pressure. The off-diagonal spatial elements T^{ij} $(i \neq j)$ represent shear stresses, sometime referred to as "anisotropic pressure".

[**Stress-energy of a perfect fluid**] A perfect fluid is specified by only two quantities: its rest-frame energy density ρ and its isotropic rest-frame pressure p. In a local inertial frame (LIF), the stress-energy tensor for such a perfect fluid is

$$T^{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{1}$$

In a more general frame, this stress-energy tensor takes the form

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}, \tag{2}$$

which of courses reduces to the above in a LIF where $u^\mu = (1,0,0,0)$ and $g^{\mu\nu} = \eta^{\mu\nu}$. The perfect fluid may seem like an unrealistic case, but it has real applications in the real world. For instance, a gas of relativistic electrons and photons form a nearly perfect fluid with $p = (1/3)\rho$. Cold dust is like a perfect fluid with $p = 0$, which seems to describe the behavior of dark matter in our Universe. Finally, we seem to live in a Universe dominated by "dark energy", which can be thought off as a perfect fluid with $p = -\rho$, resulting in $T^{\mu\nu} = -\rho g^{\mu\nu}$. Thus, the stress-energy tensor of a perfect fluid has a lot of real-world applications.

[**Connection to the action**] The stress-energy tensor can describe *any* configuration of matter/energy present in spacetime, including that of atoms forming the computer screen on which you are reading these notes. How do we compute $T^{\mu\nu}$ for such configurations? If you know the Lagrangian (really the Lagrangian density) $\mathcal{L}$ for some physical system, you can compute its stress-energy tensor using

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{3}$$

where S is the action

$$S = \int d^4x \mathcal{L}, \tag{4}$$

g is the determinant of the metric, and δ denotes a variational derivatives. The above is often quoted as the *definition* of the stress-energy tensor. It says that, up to normalization factors, the stress-energy tensor is the rate of change of the action S with respect to the spacetime metric itself.

[**Connection to thermal distributions**] Another useful case is if you have a collection of particles with known probability distribution function $f(p)$ (such as the Fermi-Dirac or Bose-Einstein distributions) for which the stress-energy tensor is

$$T^{\mu\nu} = \int \frac{d^3\vec{p}}{p^0} \sqrt{-g}\, p^\mu p^\nu f(p). \tag{5}$$

[**Conservation of energy-momentum**] In classical physics, energy and momentum are always conserved and this should be reflected in the properties of $T^{\mu\nu}$. In curved spacetime, the corresponding statement is that the stress-energy tensor is *covariantly* conserved,

$$\nabla_\nu T^{\mu\nu} = 0, \tag{6}$$

where ∇_ν is a covariant derivative. In flat spacetime, the above will always reduce to the familiar energy and momentum conservation equations. However, in curved spacetime, you may find that your "familiar" Newtonian notion of energy/momentum conservation is violated. Don't panic. $\nabla_\nu T^{\mu\nu} = 0$ *defines* the notion of energy-momentum conservation in curved spacetime.

II. ENERGY CONDITIONS

The components of the stress-energy tensor obey certain so-called "energy conditions". These are often used in General Relativity to prove certain singularity theorems and are thus useful jargon to know. Here are some of the main ones:

- The **Null Energy Condition** (NEC) states that $T_{\mu\nu} l^\mu l^\nu \geq 0$ for all null vectors l^μ. For a perfect fluid, this implies $\rho + p \geq 0$.

- The **Weak Energy Condition** (WEC) states that $T_{\mu\nu} t^\mu t^\nu \geq 0$ for all timelike vector t^μ. For a perfect fluid, this implies $\rho \geq 0$ and $\rho + p \geq 0$, i.e. like the NEC with the extra requirement that the energy density be positive.

- The **Dominant Energy Condition** (DEC) includes the WEP, with the additional requirement that $T^{\mu\nu} t_\mu$ is not a spacelike vector (that is $T_{\mu\nu} T^\nu{}_\lambda t^\mu t^\lambda \leq 0$). For a perfect fluid, this implies that $\rho \geq |p|$. The DEC implies the WEC and the NEC.

- The **Strong Energy Condition** (SEC) states that $T_{\mu\nu} t^\mu t^\nu \geq \frac{1}{2} T^\lambda{}_\lambda t^\sigma t_\sigma$ for all timelike vector t^μ. For a perfect fluid, the implies $\rho + p \geq 0$ and $\rho + 3p \geq 0$. The SEC implies the NEC, but not the WEC.