PHYS 480/581: General Relativity
Tensors

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(Dated: February 7, 2024)

I. REVIEW OF VECTORS AND DUAL VECTORS

We have seen before that a (four-)vector is an object that lives in the tangent space T_pM of some spacetime point p of some manifold M, as illustrated in Fig. 1. Dual vectors (or one-forms) live in what we call the dual vector space (or cotangent space) T_p^*M at point p. I remind you that dual vectors are linear maps from the original vector space T_pM to the real numbers; that is, if $\boldsymbol{\omega} \in T_p^*M$ is a dual vector, it acts as

$$\boldsymbol{\omega}(a\boldsymbol{V} + b\boldsymbol{W}) = a\,\boldsymbol{\omega}(\boldsymbol{V}) + b\,\boldsymbol{\omega}(\boldsymbol{W}) \in \mathbb{R}, \tag{1}$$

where $\boldsymbol{V}$, $\boldsymbol{W}$ are vectors, and a, b are real numbers. Again, you should think of a dual vector as a "machine" that "eats" a vector and returns a real number. We have also seen that we can expand a vector in terms of a coordinate basis $\boldsymbol{V} = V^\mu \mathbf{e}_{(\mu)}$. We can of course also defined a coordinate basis for dual vectors such that $\boldsymbol{\omega} = \omega_\mu \mathbf{e}^{(\mu)}$. The dual basis vector $\mathbf{e}^{(\mu)}$ are defined from the coordinate basis vectors as

$$\mathbf{e}^{(\nu)}(\mathbf{e}_{(\mu)}) = \delta^\nu_\mu. \tag{2}$$

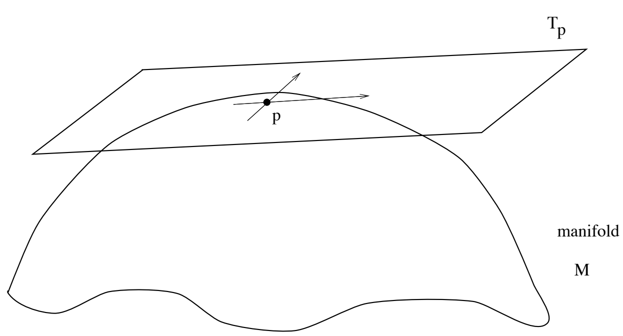

FIG. 1. Tangent space T_p of a two-dimensional manifold at point p. Reproduced from arXiv:gr-qc/9712019.

With this construction, the action of a dual vector on a vector is something we are quite familiar with

$$\begin{aligned}\boldsymbol{\omega}(\boldsymbol{V}) &= \omega_\mu \mathbf{e}^{(\mu)}(V^\nu \mathbf{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \mathbf{e}^{(\mu)}(\mathbf{e}_{(\nu)}) \\ &= \omega_\mu V^\nu \delta^\mu_\nu \\ &= \omega_\mu V^\mu,\end{aligned} \tag{3}$$

which is what we have been calling the inner (or dot) product between dual vector and a vector $\boldsymbol{\omega}\cdot\boldsymbol{V}$. Note that this product is the same in every frame. A simple example of a dual vector is the gradient of a scalar function ϕ, which we will denote $\mathrm{d}\phi$

$$\mathrm{d}\phi = (\partial_\mu \phi)\mathbf{e}^{(\mu)}, \tag{4}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. Upon a change of coordinates, the components of vectors and dual vectors transform as

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu, \qquad \omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\nu, \tag{5}$$

respectively. The corresponding transformation laws for the coordinate basis vectors and dual vectors are

$$\mathbf{e}'_{(\mu)} = \frac{\partial x^\nu}{\partial x'^\mu}\mathbf{e}_{(\nu)}, \qquad \mathbf{e}'^{(\mu)} = \frac{\partial x'^\mu}{\partial x^\nu}\mathbf{e}^{(\nu)}. \tag{6}$$

II. TENSORS

[**What is a tensor?**] Much like a dual vector is a linear map from vectors to $\mathbb{R}$, a tensor $\boldsymbol{T}$ of rank (k, l) is a multilinear map from k dual vectors and l vectors to the real numbers

$$\boldsymbol{T} : \underbrace{T_p^* \times \ldots \times T_p^*}_{k \text{ times}} \times \underbrace{T_p \times \ldots \times T_p}_{l \text{ times}} \to \mathbb{R}, \tag{7}$$

where $\times$ is the standard Cartesian product. You should think of the tensor $\boldsymbol{T}$ has a "machine" that "eats" k dual vectors and l vectors and returns a real number.

[**Tensors act linearly on their arguments**] As the "multilinear" characteristic implies, tensors act linearly on their arguments. For instance, for a rank $(1, 1)$ tensor $\boldsymbol{T}$, we have

$$\boldsymbol{T}(a\boldsymbol{\omega} + b\boldsymbol{\eta}, c\boldsymbol{V} + d\boldsymbol{W}) = ac\,\boldsymbol{T}(\boldsymbol{\omega}, \boldsymbol{V}) + ad\,\boldsymbol{T}(\boldsymbol{\omega}, \boldsymbol{W}) + bc\,\boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{V}) + bd\,\boldsymbol{T}(\boldsymbol{\eta}, \boldsymbol{W}) \in \mathbb{R}, \tag{8}$$

where a, b, c, d are real numbers, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$ are dual vectors, and $\boldsymbol{V}$, $\boldsymbol{W}$ are vectors.

[**Components of Tensors**] Now, just like we can write vectors and dual vectors in terms of their components in a coordinate basis, we can write a rank (k, l) tensor in terms of its components is such a basis. To construct a basis for the space of all (k, l) tensors, we simply take the tensor product $\otimes$ of k coordinate basis vectors $\mathbf{e}_{(\mu)}$ and l coordinate basis dual vectors $\mathbf{e}^{(\nu)}$

$$\mathbf{e}_{(\mu_1)} \otimes \ldots \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \ldots \otimes \mathbf{e}^{(\nu_l)}. \tag{9}$$

With this basis, I can write my rank (k, l) tensor $\boldsymbol{T}$ in terms of its components as

$$\boldsymbol{T} = T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \mathbf{e}_{(\mu_1)} \otimes \ldots \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \ldots \otimes \mathbf{e}^{(\nu_l)}. \tag{10}$$

[**Transformation law for tensors**] Now, the abstract tensor $\boldsymbol{T}$ must be the same in every frames. From Eq. (6) above, I know how my coordinate basis vectors and dual vectors transform under a change of coordinates. We thus have

$$\begin{aligned}
\boldsymbol{T} &= T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \mathbf{e}_{(\mu_1)} \otimes \ldots \mathbf{e}_{(\mu_k)} \otimes \mathbf{e}^{(\nu_1)} \otimes \ldots \otimes \mathbf{e}^{(\nu_l)} \\
&= T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \mathbf{e}'_{(\mu_1)} \otimes \ldots \mathbf{e}'_{(\mu_k)} \otimes \mathbf{e}'^{(\nu_1)} \otimes \ldots \otimes \mathbf{e}'^{(\nu_l)} \\
&= T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \mathbf{e}_{(\alpha_1)} \otimes \ldots \otimes \frac{\partial x^{\alpha_k}}{\partial x'^{\mu_k}} \mathbf{e}_{(\alpha_k)} \otimes \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \mathbf{e}^{(\beta_1)} \otimes \ldots \otimes \frac{\partial x'^{\nu_l}}{\partial x^{\beta_l}} \mathbf{e}^{(\beta_l)},
\end{aligned} \tag{11}$$

from which we get

$$T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l} = \frac{\partial x^{\alpha_1}}{\partial x'^{\mu_1}} \cdots \frac{\partial x^{\alpha_k}}{\partial x'^{\mu_k}} \frac{\partial x'^{\nu_1}}{\partial x^{\beta_1}} \cdots \frac{\partial x'^{\nu_l}}{\partial x^{\beta_l}} T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l}. \tag{12}$$

Inverting this relationship, we get

$$T'^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_l}}{\partial x'^{\nu_l}} T^{\alpha_1 \ldots \alpha_k}{}_{\beta_1 \ldots \beta_l}. \tag{13}$$

Equation (13) is the *definition* of a rank (k, l) tensor. Any object that transforms like in Eq. (13) under a coordinate transformation $x \to x'$ will be a bone fide rank (k, l) tensor. Note: not every object having k upper indices and l lower indices is a tensor. We will see some examples of this later on.

[**How tensors act on their arguments**] Now that we know how component of tensors can be written down in coordinate bases, we can easily understand how a tensor $\boldsymbol{T}$ acts on its arguments. For instance, for a rank $(1, 2)$ tensor that takes in two vectors $\boldsymbol{V}$, $\boldsymbol{W}$ and one dual vectors $\boldsymbol{\omega}$ to return a real number, we have

$$\begin{aligned}
\boldsymbol{T}(\boldsymbol{\omega}, \boldsymbol{V}, \boldsymbol{W}) &= T^{\mu}{}_{\nu\sigma} \mathbf{e}_{(\mu)} \otimes \mathbf{e}^{(\nu)} \otimes \mathbf{e}^{(\sigma)} (\omega_\alpha \mathbf{e}^{(\alpha)}, V^\beta \mathbf{e}_{(\beta)}, W^\gamma \mathbf{e}_{(\gamma)}) \\
&= T^{\mu}{}_{\nu\sigma} \mathbf{e}_{(\mu)} (\omega_\alpha \mathbf{e}^{(\alpha)}) \otimes \mathbf{e}^{(\nu)} (V^\beta \mathbf{e}_{(\beta)}) \otimes \mathbf{e}^{(\sigma)} (W^\gamma \mathbf{e}_{(\gamma)}) \\
&= T^{\mu}{}_{\nu\sigma} \omega_\alpha V^\beta W^\gamma \delta^\alpha_\mu \delta^\nu_\beta \delta^\sigma_\gamma \\
&= T^{\mu}{}_{\nu\sigma} \omega_\mu V^\nu W^\sigma \in \Re,
\end{aligned} \tag{14}$$

which is a rather intuitive way for tensors to act on vectors and dual vectors.

[**No need to act on all their arguments**] While we have defined tensors as multilinear maps from a set of vectors and dual vectors to the real numbers, we do not always have to act the tensor on its full set of arguments. For instance, for our rank $(1,2)$ tensor $\boldsymbol{T}$ above, we can act it on a single vector $\boldsymbol{V}$ to form a perfectly fine rank $(1,1)$ tensor $\boldsymbol{S}$, whose components are given by

$$S^{\mu}{}_{\nu} = T^{\mu}{}_{\nu\sigma} V^{\sigma}. \tag{15}$$

I could also act the tensor $\boldsymbol{T}$ on a rank $(2,0)$ tensor $\boldsymbol{R}$ to form a perfectly fine vector $\boldsymbol{U}$

$$U^{\mu} = T^{\mu}{}_{\nu\sigma} R^{\nu\sigma}. \tag{16}$$

III. TENSORS OPERATIONS

A. Trace

The trace X of a $(1,1)$ tensor $\boldsymbol{X}$ is

$$X = X^{\nu}{}_{\nu}. \tag{17}$$

If you think of $X^{\nu}{}_{\mu}$ has a matrix, then this is just the sum of the diagonal elements. However, if you are given $X_{\mu\nu}$, than X is not the sum of the diagonal elements of $X_{\mu\nu}$ since

$$X = X^{\nu}{}_{\nu} = g^{\nu\alpha} X_{\alpha\nu}. \tag{18}$$

A good example of this the trace of the Minkowski metric $\eta_{\mu\nu}$. We might naively think that its trace is $-1+1+1+1 = 2$, while in fact it is

$$\eta^{\mu\nu}\eta_{\mu\nu} = \delta^{\mu}_{\mu} = 4. \tag{19}$$

B. Contraction (or partial tracing)

We can contract one upper and one lower indices from a rank (k,l) tensor to form a $(k-1,l-1)$ tensor

$$S^{\mu\rho}{}_{\sigma} = T^{\mu\nu\rho}{}_{\sigma\nu}. \tag{20}$$

In general, which indices are contracted together is important; different choices will lead to different results, i.e.

$$T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}. \tag{21}$$

C. Addition

You can add two rank (k,l) tensors to form another rank (k,l) tensor

$$T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} = S^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l} + U^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}. \tag{22}$$

D. Product

We can create a rank (k_1+k_2, l_1+l_2) tensor by taking the tensor product between a rank (k_1,l_1) tensor and a rank (k_2,l_2) tensor

$$T^{\mu_1\ldots\mu_{k_1+k_2}}{}_{\nu_1\ldots\nu_{l_1+l_2}} = S^{\mu_1\ldots\mu_{k_1}}{}_{\nu_1\ldots\nu_{l_1}} \otimes U^{\mu_{k_1+1}\ldots\mu_{k_2}}{}_{\nu_{l_1+1}\ldots\nu_{l_2}}. \tag{23}$$

E. Lowering or raising indices

We can use the metric $g_{\mu\nu}$ and inverse metric $g^{\mu\nu}$ to raise or lower indices, i.e.

$$T^{\mu\nu\rho}{}_{\sigma\nu} = g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g_{\sigma\delta} g_{\nu\phi} T_{\alpha\beta\gamma}{}^{\delta\phi}. \tag{24}$$

F. Symmetric and antisymmetric tensors

A tensor is said to be symmetric in two of its first and third indices if

$$S_{\mu\rho\nu} = S_{\nu\rho\mu}. \tag{25}$$

Similarly, a tensor is said to be symmetric in its two first indices if

$$S_{\mu\rho\nu} = S_{\rho\mu\nu}. \tag{26}$$

A tensor is to be symmetric if it is unchanged under all possible permutations of its indices.

$$S_{\mu\rho\nu} = S_{\mu\nu\rho} = S_{\rho\mu\nu} = S_{\rho\nu\mu} = S_{\nu\rho\mu} = S_{\nu\mu\rho}. \tag{27}$$

For instance, the metric is a symmetric $(0, 2)$ tensor since $g_{\mu\nu} = g_{\nu\mu}$. A tensor is said to be antisymmetric in two of its first and third indices if

$$S_{\mu\rho\nu} = -S_{\nu\rho\mu}. \tag{28}$$

Similarly, a tensor is said to be antisymmetric in its two first indices if

$$S_{\mu\rho\nu} = -S_{\rho\mu\nu}. \tag{29}$$

If a tensor is said to be antisymmetric in all its indices, we just call this tensor antisymmetric (or completely antisymmetric).

G. Partial Derivatives

In general, the partial derivative of a tensor is **not** a tensor. For instance,

$$T_{\alpha}{}^{\mu}{}_{\nu} = \partial_{\alpha} R^{\mu}{}_{\nu} \tag{30}$$

is not a tensor in a general spacetime (however, it is in flat spacetime). We will soon see how to generalize the notion of derivative in curved spacetime.