PHYS 480/581
General Relativity

Midterm Exam Solutions
Wednesday 03/06/2024

Question 1 (20 points).

Consider the two-dimensional metric

$$ds^2 = -e^{2gx}dt^2 + dx^2, \tag{1}$$

where g is a real constant.

(a) Compute the 6 independent Christoffel connection coefficients $\Gamma^\alpha_{\mu\nu}$.
Solutions:
The only nonzero derivative of the metric is $\partial_x g_{tt} = -2ge^{2gx}$. Since the metric is diagonal, only Christoffels involving two t and one x indices will thus be nonzero, implying $\Gamma^t_{tt} = \Gamma^x_{xx} = \Gamma^x_{xt} = \Gamma^t_{xx} = 0$. The others are

$$\begin{aligned}\Gamma^t_{tx} = \Gamma^t_{xt} &= \frac{1}{2}g^{tt}\left(\partial_t g_{xt} + \partial_x g_{tt} - \partial_t g_{tx}\right)\\ &= \frac{1}{2}g^{tt}\partial_x g_{tt}\\ &= -\frac{1}{2}e^{-2gx}(-2ge^{2gx})\\ &= g,\end{aligned} \tag{2}$$

and

$$\begin{aligned}\Gamma^x_{tt} &= \frac{1}{2}g^{xx}\left(\partial_t g_{tx} + \partial_t g_{xt} - \partial_x g_{tt}\right)\\ &= -\frac{1}{2}g^{xx}\partial_x g_{tt}\\ &= -\frac{1}{2}(-2ge^{2gx})\\ &= ge^{2gx}.\end{aligned} \tag{3}$$

(b) Use the geodesic equation to show that at every point in this spacetime, a free particle initially at rest will experience an initial acceleration in the x direction given by

$$\frac{d^2x}{d\tau^2} = -g. \tag{4}$$

[Note that a particle initially at rest has $dx/d\tau = 0$. Also use the normalization $u_\mu u^\mu = -1$, where $u^\mu = dx^\mu/d\tau$.]

Solutions:
Consider the x component of the geodesic equation

$$\frac{d^2x}{d\tau^2} + \Gamma^x_{tt}\left(\frac{dt}{d\tau}\right)^2 = \frac{d^2x}{d\tau^2} + ge^{2gx}\left(\frac{dt}{d\tau}\right)^2 = 0, \tag{5}$$

where we have used the only Christoffel connection with an upper x index that is nonzero. Now, consider the normalization of the "four"-velocity

$$\begin{aligned} g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} &= g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{xx}\left(\frac{dx}{d\tau}\right)^2 \\ &= -e^{2gx}\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 \\ &= -1. \end{aligned} \tag{6}$$

Now for a particle initially at rest we have $dx/d\tau = 0$, and the above implies that

$$e^{2gx}\left(\frac{dt}{d\tau}\right)^2 = 1. \tag{7}$$

Substituting this in Eq. (5) above, we have

$$\frac{d^2x}{d\tau^2} = -g, \tag{8}$$

which is the desired result.

(c) Compute the one independent component of the Riemann tensor. Is this spacetime curved?
Solutions:
The symmetry property of the Riemann tensor ensures that the only nonzero components are those involving two t and two x indices. Of those, only the components for which the two first indices *and* the two last indices are different will be nonzero. This leaves

$$\begin{aligned} R^t{}_{xtx} &= \partial_t\Gamma^t_{xx} - \partial_x\Gamma^t_{xt} + \Gamma^t_{t\beta}\Gamma^\beta_{xx} - \Gamma^t_{x\beta}\Gamma^\beta_{xt} \\ &= -\left(\partial_x\Gamma^t_{xt} + \Gamma^t_{x\beta}\Gamma^\beta_{xt}\right) \\ &= -\left(\partial_x g + \Gamma^t_{xt}\Gamma^t_{xt}\right) \\ &= -g^2. \end{aligned} \tag{9}$$

Since the Riemann tensor is nonzero, this space is indeed curved.

(d) Show that this spacetime has constant scalar (Ricci) curvature R everywhere.
Solutions:
The Ricci tensor is

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{10}$$

We thus have

$$\begin{aligned} R_{tt} &= R^{\alpha}{}_{t\alpha t} \\ &= R^{x}{}_{txt} \\ &= g^{xx} R_{xtxt} \\ &= g^{xx} R_{txtx} \\ &= g^{xx} g_{tt} R^{t}{}_{xtx} \\ &= g^2 e^{2gx}. \end{aligned} \tag{11}$$

$$\begin{aligned} R_{xx} &= R^{\alpha}{}_{x\alpha x} \\ &= R^{t}{}_{xtx} \\ &= -g^2. \end{aligned} \tag{12}$$

$$\begin{aligned} R_{xt} &= R^{\alpha}{}_{x\alpha t} \\ &= 0. \end{aligned} \tag{13}$$

The scalar curvature is thus

$$\begin{aligned} R &= g^{\mu\nu} R_{\mu\nu} \\ &= g^{tt} R_{tt} + g^{xx} R_{xx} \\ &= -e^{-2gx} g^2 e^{2gx} - g^2 \\ &= -2g^2, \end{aligned} \tag{14}$$

which is constant everywhere.

Question 2 (5 points).

An astronaut wakes up in a small spaceship with no previous memory of how they got there. The spaceship has no window and both the onboard computer and engine are broken, so the astronaut has no idea where they are in the Universe. The astronaut feels weightless but they do not know whether (a) the spaceship is moving at constant velocity through empty space or (b) freely-falling towards a massive celestial object. The astronaut has access to a bag of glass marbles and a precise ruler.

Design a simple experiment that the astronaut can do to determine whether they are in situation (a) or (b). Make a sketch of your proposed experimental setup. Justify your choice of experiment with the relevant equation(s).

Solutions:
From the Equivalence Principle, it seems at first that there is nothing that the astronaut could do to distinguish between the two situations. However, the Equivalence Principle is only valid in a small region of spacetime. Since the spaceship is large enough to host an astronaut, this is more than sufficient to build an experiment that can distinguish between (a) and (b). In situation (a)

the spacetime is flat and geodesics that are initially parallel will remain parallel forever. In terms of geodesic deviation, we have

$$\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha = 0, \tag{15}$$

where $\mathbf{n} = n^\nu \mathbf{e}_{(\nu)}$ is a vector separating two geodesics. In situation (b), near a massive celestial object, spacetime will be curved ($R^\alpha{}_{\mu\nu\sigma} \neq 0$), and geodesics that are initially parallel will start to converge (or diverge) towards each other according to

$$\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha = -R^\alpha{}_{\mu\nu\sigma} u^\sigma u^\mu n^\nu. \tag{16}$$

Thus if the astronaut places two marbles at rest (in the rest frame of the spaceship) separated by a macroscopic distance measurable with the ruler (and not touching the walls of the spaceship), and then wait for a certain amount of time and then measure the separation between the marbles again. If the distance between the marbles has changed, then the astronaut would know that they are in the vicinity of a massive celestial object. Otherwise, they are in empty space.

Another way to explain this would be to think that the gravitational field near a massive object is nonuniform (corresponding to $R^\alpha{}_{\mu\nu\sigma} \neq 0$), which will cause the two marbles to accelerate with respect to each other, hence changing their separation.

Useful Equations

Proper time

$$d\tau = \sqrt{-ds^2}. \tag{17}$$

Covariant derivative of a vector and dual vector

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho, \qquad \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho. \tag{18}$$

Christoffel connection coefficients

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}). \tag{19}$$

The geodesic equation

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0. \tag{20}$$

The Riemann tensor

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\gamma}\Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\beta\mu}. \tag{21}$$

The Ricci tensor and scalar

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}. \tag{22}$$

Geodesic deviation equation

$$\left(\frac{d^2\mathbf{n}}{d\tau^2}\right)^\alpha = -R^\alpha{}_{\mu\nu\sigma} u^\sigma u^\mu n^\nu, \tag{23}$$

where $\mathbf{n} = n^\nu \mathbf{e}_{(\nu)}$ is a vector separating two geodesics, and $u^\mu = dx^\mu/d\tau$.